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『Homogeneity, Saddle Path Stability, and Logarithmic Preferences in Economic Models』

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Homogeneity, Saddle Path Stability, and Logarithmic Preferences in Economic Models*

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Summary. In a stylized Robinson Crusoe economy, we demonstrate the usefulness of homogeneity in initial conditions when solving and analyzing macroeconomic models. In a first step, we define state-like and control-like variables. In a second step, we introduce the value-function-like function. While the former step reduces the number of variables that have to be considered when solving the model, the latter step reduces the dimensionality of the Bellman equation associated with the optimization problem. The model's solution is shown to be saddle-path stable, such that the phase diagram associated with the Bellman equation has two solution branches and the structure of our model allows us to state both the stable and the unstable branch explicitly. We also explain the usefulness of logarithmic preferences when studying the continuous-time Hamilton-Jacobi-Bellman equation. In this case the utility maximization problem can be transformed into an initial value problem for an ordinary differential equation.
Keywords: Closed-form solution, saddle path, homogeneity in initial conditions, continuous time.
JEL Classifications: C61, C65, C63.

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1. Introduction

Solving and analyzing models is a key competence in the profession of macroeconomics. An example for a very common computational approach is local linearization around the steady state or around the balanced growth path. The mechanical application of such standard strategies, however, often leads researchers to ignore some mathematical properties that are not only typical for macroeconomic models but also useful when it comes to their analysis. As a consequence, economists often have merely a vague idea of these properties and the underlying mathematics and might therefore miss opportunities to apply more powerful tools. In this paper, we show that this is the case for *homogeneity in initial conditions*, *saddle path stability*, and *logarithmic preferences*. These properties allow to apply both simple procedures to reduce the complexity of a problem, and easy to implement numerical schemes that are not restricted to the neighbourhood of the steady state. Within the framework of a 'Robinson Crusoe economy', a very stylized version of the two sector Uzawa [13] and Lucas [7] model of endogenous growth, we show how these properties are related to the introduction of so-called 'state-like' and 'control-like' variables (cf. Mulligan and Sala-i-Martin [9]). This model thus allows us to illustrate a simple solution method that uses the 'value-function-like function' and leads to a numerical problem that can be solved easily: an initial value problem for an ordinary differential equation.

We use the value function approach to solve the model. Based on the guess-and-verify method, we find two functions that solve Crusoe's Bellman equation. Both solutions are candidates for his value function. After studying both functions in detail, one of these candidates is found to be the unstable solution in the phase diagram, while the other is indeed Crusoe's value function. Indeed, the transversality conditions provide the criterion to discriminate between the two solutions. Furthermore, we show that both functions share a common fixed point. This point is attracting on the stable branch and repelling on the unstable branch: the saddle point property.

Using homogeneity in the initial conditions, a property that applies to most economic models, we transform the model's variables and introduce a state-like and a control-like variable, which reduces the number of variables that have to be considered when analyzing our stylized model. A similar transformation can also be applied to the value function and we are led to the value-function-like function. This last step sheds new light on logarithmic utility functions. For in this case the shadow values of the two capital stocks - which correspond to the derivatives of the value function - can be stated in terms of the derivative of the value-function-like function only, a property that paves the way for the simple solution method mentioned above. When dealing with the more general isoelastic utility function, the value-function-like function itself enters the shadow values. In contrast to the logarithmic utility case, therefore, more sophisticated solution methods have to be applied in this case.

The model is stylized in the sense of simplicity. Throughout the paper, we assume a deterministic setup without externalities. By following McCallum [8] for the (discrete time) neoclassical stochastic growth model, we assume logarithmic preferences together with full depreciation of the capital stocks. In order to examine the usefulness of the 'value-function-like function', a continuous time version of the model is analyzed in the last part of the paper (cf. Section 6). There we abstract from any depreciation of the capital stocks. For the main body of the paper, the simplicity of the Robinson Crusoe economy allows us to use explicit functional forms such that the mathematics can be kept relatively simple and the ideas presented are more intuitive, although they are also applicable when relying on numerical solution methods in more complicated models. Applications of our findings, therefore, are not restricted to models of similar simplicity, on the contrary. Our model choice is due to illustrative purposes, such that it easily allows to transfer the proposed strategy to more general problems.[1] In the spirit of Occam's razor, we make no more assumptions than are necessary to illustrate our ideas. Readers that are interested in the dynamics of models of the Uzawa-Lucas type are referred to the already rich and still growing literature (cf. Bond et al. [2], Caballé and Santos [3], Ladrón-de-Guevara et al. [4] and [5], Mulligan and Sala-i-Martin [9], Ortigueira and Santos [10], Xie [14]).

The paper is organized as follows. Section 2 introduces the Robinson Crusoe economy. Section 3 presents Crusoe's dynamic optimization problem and characterizes its solution. Based on the two solutions to the Bellman equation and on the introduction of the state-like and control-like variable, Section 4 discusses the saddle path property analytically and graphically. Section 5 defines the value-function-like function. Section 6 considers a continuous time version of the model and presents the simple solution method. We also explain why this method depends on the logarithmic utility assumption. Section 7 concludes.

2. The model

Consider a closed economy populated by an arbitrary number of identical and infinitely-lived agents. The representative agent, Robinson Crusoe, enters every period with predetermined endowments of human and physical capital, denoted by h_t and k_t, respectively. There are two sectors in the economy. Firms produce a single homogeneous good and a schooling sector produces education. Both sectors use constant returns to scale technologies in the reproducible factors. We assume that the population is constant over time and normalized to unity.

Crusoe has logarithmic preferences with respect to sequences of consumption:

$$U(c_0, c_1, ..) = \sum_{t=0}^{\infty} \beta^t \ln(c_t), \tag{1}$$

where c_t is the level of consumption in period $t \in \mathbb{N}_0$ and $\beta \in (0,1)$ is the subjective discount factor. Note that the logarithmic utility function equation implies that the intertemporal elasticity of substitution is equal to one.

[1] Including external effects of human capital, Bethmann [1] applies these ideas to an N player differential game.

In each period, Crusoe has a fixed endowment of time, which is normalized to unity. As he does not benefit from leisure, he allocates his whole time budget to the two production sectors. The variable u_t denotes the fraction of time allocated to goods production in period t. The remainder, i.e. the fraction $1-u_t$, is spent in the schooling sector. The production of human capital is linear in both time and human capital:

$$h_{t+1} = B\left(1-u_t\right)h_t, \qquad \forall t \in \mathbb{N}_0,\ h_0 > 0, \tag{2}$$

where parameter B is assumed to be positive. The schooling technology implies that both the realized marginal and the realized average product of human capital are equal to $B\left(1-u_t\right)$. Note that we assume full depreciation of human capital.

The consumption good is produced by a large number of identical firms. Each of these firms employs the same Cobb-Douglas technology in physical capital k_t and effective work $h_t u_t$.[2] Hence, output y_t of the representative firm is given by:

$$y_t = Ak_t^{\alpha}\left(u_t h_t\right)^{1-\alpha}. \tag{3}$$

The parameter α is the output elasticity of physical capital and we assume $\alpha \in (0,1)$. The parameter A denotes the constant total factor productivity. Since full depreciation of physical capital is assumed, Crusoe's constraint associated with his stock of physical capital is given by:

$$y_t = c_t + k_{t+1}, \qquad \forall t \in \mathbb{N}_0,\ k_0 > 0. \tag{4}$$

Having completed the presentation of the model, the next section will state Crusoe's dynamic optimization problem (DOP) and then apply the value function approach to characterize its solution.

3. Dynamic optimization problem

Robinson Crusoe chooses the paths $\{c_t\}_{t=0}^{\infty}$ and $\{u_t\}_{t=0}^{\infty}$ to maximize (1) subject to the constraints that are associated with his capital stocks, (2) and (4 with 3). In equilibrium, the constraints lead to transversality conditions which ensure that he obeys his intertemporal budget constraints. Crusoe maximizes his objective function:

$$\max_{\{c_t,u_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln\left(c_t\right), \tag{5}$$

[2] The case where effective work plays no role in goods production corresponds to the neoclassical growth model.

subject to the state dynamics, control constraints, and initial conditions:

$$
\begin{aligned}
k_{t+1} &= Ak_t^{\alpha} u_t^{1-\alpha} h_t^{1-\alpha} - c_t, && \forall\, t \in \mathbb{N}_0, && (6)\\
h_{t+1} &= B\left(1-u_t\right) h_t, && \forall\, t \in \mathbb{N}_0, && (7)\\
c_t &\geq 0 \quad \text{and} \quad 1 \geq u_t \geq 0, && \forall\, t \in \mathbb{N}, && (8)\\
k_t &\geq 0 \quad \text{and} \quad h_t \geq 0, && \forall\, t \in \mathbb{N}, && (9)\\
k_0 &> 0 \quad \text{and} \quad h_0 > 0. && && (10)
\end{aligned}
$$

Note that the logarithmic utility function penalizes zero consumption. This fact, together with the Cobb-Douglas technology in goods production, ensures that both factor inputs are positive for all $t \in \mathbb{N}_0$. Requiring the initial stocks of capital to be strictly positive ensures an interior solution and rules out trivial solutions, where $k_\tau = 0$ or $h_\tau = 0$ hold for some $0 < \tau < \infty$. Hence, we know that the sequences of optimal controls $\{c_t^*\}_{t=0}^{\infty}$ and $\{u_t^*\}_{t=0}^{\infty}$ chosen by Robinson Crusoe satisfy:

$$
c_t^* > 0 \qquad \text{and} \qquad 0 < u_t^* < 1, \tag{11}
$$

for all $t \in \mathbb{N}_0$. The solution to Crusoe's DOP from period t onwards:

$$
V\left(k_t, h_t\right) = \max_{\{c_s, u_s\}_{s=t}^{\infty}} \sum_{s=t}^{\infty} \beta^{s-t} \ln\left(c_s\right) \quad \text{s.t. (6) and (7)} \tag{12}
$$

is the unique value function V, that also solves the following Bellman equation associated to the above DOP (cf. Stokey and Lucas [12], Section 4.4):

$$
V\left(k_t, h_t\right) = \max_{c_t, u_t} \left\{\ln\left(c_t\right) + \beta V\left(k_{t+1}, h_{t+1}\right)\right\}. \tag{13}
$$

However, unboundedness of logarithmic preferences implies that a solution to this Bellman equation is not necessarily the unique value function. Using the differentiability of the value function[3], we derive for each t the first-order necessary conditions for the optimal consumption choice c_t^*, and for the optimal fraction of time allocated to human capital formation u_t^*:

$$
\begin{aligned}
c_t^* &= \frac{1}{\beta \frac{\partial V_{t+1}}{\partial k_{t+1}}}, && (14)\\
u_t^* &= \left(\frac{\frac{\partial V_{t+1}}{\partial k_{t+1}}(1-\alpha)A}{\frac{\partial V_{t+1}}{\partial h_{t+1}} B}\right)^{\frac{1}{\alpha}} \frac{k_t}{h_t}, && (15)
\end{aligned}
$$

where we use the notation V_{t+1} as an abbreviation for $V(k_{t+1}, h_{t+1})$.[4] Condition (14) characterizes the effect of shifting one unit of today's output from consumption to in-

[3] A proof of the differentiability is given, for example, in Stokey and Lucas [12], Chapter 4. They also show that the Bellman equation may have more than one solution in the unbounded returns case.

[4] Note, that V is not time dependent.

vestment along the optimal decision path. Today's marginal change in utility equals the discounted marginal change in tomorrow's wealth with respect to tomorrow's capital stock. Equation (15) is solved for the optimal allocation of human capital between the two sectors and seems therefore byzantine. However it simply states that u_t^* balances the weighted marginal changes in goods production and human capital formation that are due to a shifting of a marginal unit of human capital from one sector to the other. The respective weights are the derivatives of the value function with respect to the corresponding state variable.[5]

The next section establishes that there are indeed two functions that solve the above Bellman equation (13) so that we have two candidates for the value function. We derive the controls implied by both functions in order to verify that both functions satisfy the first-order necessary conditions as well as the respective Euler equations. Only the transversality conditions show that one of these candidates cannot be the true value function.

4. Saddle path property

4.1. Two candidates for the unique value function

The value function is a differentiable function that satisfies necessarily the Bellman equation (13) and the first-order necessary conditions (14) and (15). In this section we argue that there are exactly two functions satisfying these requirements. Indeed we state both functions explicitly:

$$V\left(k_t, h_t\right) = \theta + \theta_k \ln k_t + \theta_h \ln h_t, \tag{16}$$
$$W\left(k_t, h_t\right) = \varrho + \varrho_1 \ln\left[k_t + \varrho_2 h_t\right], \tag{17}$$

where the parameters θ_k, θ_h, ϱ_1, and ϱ_2 are defined as follows:[6]

$$\theta_k = \tfrac{\alpha}{1-\alpha\beta}, \quad \theta_h = \tfrac{1-\alpha}{(1-\beta)(1-\alpha\beta)}, \quad \varrho_1 = \tfrac{1}{1-\beta}, \quad \varrho_2 = (1-\alpha)\left(\tfrac{\alpha^\alpha A}{B}\right)^{\frac{1}{1-\alpha}}. \tag{18}$$

The controls implied by the first-order necessary conditions (14) and (15) when using the first solution (16) to the Bellman equation are given by:

$$c_{V,t}^* = (1-\alpha\beta)\, y_t \qquad \text{and} \qquad u_{V,t}^* = 1-\beta. \tag{19}$$

When using the solution (17) we obtain:

$$c_{W,t}^* = (1-\beta)\, Bk_t + (1-\beta)\, B\varrho_2 h_t \quad \text{and} \quad u_{W,t}^* = \left(\tfrac{B}{\alpha A}\right)^{\frac{1}{1-\alpha}} \tfrac{k_t}{h_t}. \tag{20}$$

[5] Note, that these derivatives can be interpreted as shadow prices.

[6] The constant terms ϱ and θ are given by: $\varrho := \frac{\ln[1-\beta]}{1-\beta} + \frac{\beta \ln \beta}{(1-\beta)^2} + \frac{\ln B}{(1-\beta)^2}$ and $\theta := \frac{\ln A}{(1-\beta)(1-\alpha\beta)} + \frac{(1-\alpha)\beta \ln B}{(1-\beta)^2(1-\alpha\beta)} + \ln[1-\alpha\beta] + \frac{(1-\alpha)\ln[1-\beta]}{1-\alpha\beta} + \frac{\alpha\beta\ln\alpha}{1-\alpha\beta} + \frac{\beta\ln\beta}{1-\beta}$.

Both functions (16) and (17) are candidates for the value function associated with Crusoe's optimization problem. A first naive attempt to discriminate between these candidates is to check whether one of them actually minimizes the expression inside the braces in the Bellman equation (13). The following proposition ensures that the above first-order necessary conditions are indeed sufficient for maximizing the right-hand side of (13).

Proposition 1 (Sufficiency) *The controls* (14) *and* (15) *are necessary and sufficient for maximizing the Bellman equation under the functions V and W.*

The appendix provides a proof of Proposition 1. The next step is to check whether the Euler equations help to decide which of the implied controls should be chosen. Using the envelope property of the optimal decision rules:

$$c_t^* = c\left(k_t, h_t\right) \qquad \text{and} \qquad u_t^* = u\left(k_t, h_t\right), \tag{21}$$

leads to the following Euler equations along the optimal control paths:

$$\frac{1}{c_t^*} = \frac{\beta}{c_{t+1}^*}\frac{\alpha y_{t+1}}{k_{t+1}}, \tag{22}$$

$$u_t^* = \left(\frac{u_{t+1}^* h_{t+1}}{k_{t+1}}\frac{\alpha A}{B}\right)^{\frac{1}{\alpha}}\frac{k_t}{h_t}. \tag{23}$$

Using the function V, it is easy to see that the controls derived above satisfy these conditions. In order to verify that the controls implied by W also fulfill the Euler equations, note that:

$$y_{W,t+1} = \frac{B}{\alpha}k_{W,t+1}, \quad c_{W,t+1}^* = B\beta c_{W,t}^*, \quad u_{W,t+1}^* = \left(\frac{B}{\alpha A}\right)^{\frac{1}{1-\alpha}}\frac{k_{W,t+1}}{h_{W,t+1}}, \tag{24}$$

hold under W. Hence, we are still not able to decide which controls are the 'right' ones. Both Euler equations are necessary for a policy to attain the optimum. Together with the following transversality conditions, they are also sufficient (cf. Stokey and Lucas [12], Section 4.4):

$$\lim_{T\to\infty}\beta^T\frac{1}{c_T}\frac{\alpha y_T}{k_T}k_T = 0 \qquad \text{and} \qquad \lim_{T\to\infty}\beta^T\frac{1}{c_T}\frac{(1-\alpha)\,y_T}{u_T h_T}h_T = 0. \tag{25}$$

The transversality conditions tell us that the discounted marginal utility of an additional unit of 'last period's' capital stock is equal to zero. These conditions, therefore, guarantee that a central planner obeys his intertemporal budget constraints. In decentralized economies, these requirements rule out Ponzi games and are therefore sometimes called 'no chain letter conditions'(cf. Lerner [6]).

As long as $k_0 > 0$ and $h_0 > 0$ hold, the controls implied by V always satisfy the above transversality conditions (25). The laws of motion for physical and human capital under

V are given by the following system of equations:

$$\begin{aligned} k_{V,t+1} &= \alpha\beta(1-\beta)^{1-\alpha} A k_{V,t}^{\alpha} h_{V,t}^{1-\alpha}, && (26)\\ h_{V,t+1} &= B\beta h_{V,t}. && (27) \end{aligned}$$

In order to show that the controls implied by W do not meet in general the transversality conditions, first note that they imply for the one-step evolution of the capital stocks:

$$\begin{aligned} k_{W,t+1} &= \tfrac{B(1-\alpha+\alpha\beta)}{\alpha} k_{W,t} - B\varrho_2 (1-\beta)\, h_{W,t}, && (28)\\ h_{W,t+1} &= -B \left(\tfrac{B}{\alpha A}\right)^{\frac{1}{1-\alpha}} k_{W,t} + B h_{W,t}. && (29) \end{aligned}$$

While the dynamics under V are described by non-linear difference equations, the evolution of the state variables under W is linear. Representing the latter case in matrix notation, leads to eigenvalues $\frac{B}{\alpha}$ and $B\beta$. In contrast to (26) and (27), the last system of equations cannot exclude the possibility that the capital stocks turn negative one day which would violate the above transversality conditions. Proposition 2 generalizes the above result for the T-step ($T \in \mathbb{N}$) laws of motion of both capital stocks.

Proposition 2 (T-step dynamics under W) *Assuming k_t and h_t to be given, the optimal controls implied by the function W lead to the following capital stocks in period $t+T$:*

$$\begin{aligned} k_{W,t+T} &= B^T \frac{(\alpha\beta)^T + (1-\alpha)\sum_{i=0}^{T-1}(\alpha\beta)^i}{\alpha^T} k_{W,t} - B^T \frac{\varrho_2 (1-\beta)\sum_{i=0}^{T-1}(\alpha\beta)^i}{\alpha^{T-1}} h_{W,t}, && (30)\\ h_{W,t+T} &= B^T \frac{(\alpha\beta)^{T-1} + (1-\beta)\sum_{i=0}^{T-2}(\alpha\beta)^i}{\alpha^{T-1}} h_{W,t} - B^T \frac{\left(\frac{B}{\alpha A}\right)^{\frac{1}{1-\alpha}}\sum_{i=0}^{T-1}(\alpha\beta)^i}{\alpha^{T-1}} k_{W,t}. && (31) \end{aligned}$$

The proof of this result is by induction. Proposition 3 states that the controls and the laws of motion of the capital stocks implied by the function W in general violate the transversality conditions, except for a very special case.

Proposition 3 (Saddle point) *As long as the positive initial values of the capital stocks satisfy:*

$$\tfrac{k_0}{h_0} = (1-\beta)\left(\tfrac{\alpha A}{B}\right)^{\frac{1}{1-\alpha}} = x_{ss}, \qquad (32)$$

the controls implied by W fulfill the transversality conditions (25). *If this restriction is not met, the controls implied by W do not solve Robinson Crusoe's DOP.*

In the following, both solutions of the Bellman equation are examined in greater detail. It is shown that x_{ss} is their common fixed point. Furthermore, we prove that the solution V converges to this point for all admissible initial states. On the other hand, the solution W diverges in general. The exception occurs when $k_0/h_0 = x_{ss}$ holds. In this case, the controls implied by V and W are the same. Before we formally prove these properties, we reduce the dimensionality of the state space by introducing a transformation that is based on the homogeneity of the model.

4.2. Homogeneity and state-like and control-like variables

To illustrate the importance of the initial values of the physical and human capital stocks k_0 and h_0, suppose that Robinson Crusoe's initial endowments are given by $\tilde{k}_0$ and $\tilde{h}_0$, with $\frac{k_0}{h_0} = \frac{\tilde{k}_0}{\tilde{h}_0}$. Then the sequence of optimal human capital allocations between the two production sectors would remain unchanged, i.e. $\tilde{u}_t = u_t$ for all $t \in \mathbb{N}_0$. This finding reflects a property of linear homogeneous production functions. The marginal products of physical and human capital in the goods sector can be expressed as functions of the k/h ratio alone. Hence the marginal rate of technical substitution does not provide an incentive to reallocate human capital when the initial endowments are varied proportionately. Furthermore, Robinson would choose $\tilde{c}_t = c_t \frac{\tilde{h}_0}{h_0}$ as the utility maximizing sequence of consumption levels. Therefore tomorrow's marginal rate of technical substitution would also remain unchanged. Recapitulating the above, one would observe the following:

$$\frac{\tilde{k}_t}{k_t} = \frac{\tilde{h}_t}{h_t} = \frac{\tilde{c}_t}{c_t} = \frac{\tilde{h}_0}{h_0} \quad \text{and} \quad \frac{\tilde{u}_t}{u_t} = 1 \quad \forall t \in \mathbb{N}_0. \tag{33}$$

This property is very useful, since it allows us to follow Mulligan and Sala-i-Martin [9] in reducing the state space and introducing the state-like variable x as the following transformation of the two capital stocks: $x_t := \frac{k_t}{h_t}$. Similarly, we can introduce the control-like variable $q_t := \frac{c_t}{h_t}$.[7] Consequently, the dynamics of the state-like variable implied by the laws of motion (6) and (7) is then described by:

$$x_{t+1} = \frac{A x_t^{\alpha} u_t^{1-\alpha} - q_t}{B\left(1 - u_t\right)}. \tag{34}$$

Note that expressing the Euler equations (22) and (23) in terms of x, q, and u and using the above law of motion would constitute a three dimensional system of equations. Without knowing the explicit solution, a loglinearized version of this system would be the natural starting point for an analysis.

To study the dynamics implied by the two solutions to the Bellman equation (13), we insert the respective controls into the law of motion (34) of the state-like variable x. First, we turn to the function W in (17). Note that the control-like variable q_t^W, i.e. the rescaled goods consumption, and the human capital allocation u_t^W implied by the function W are given by:

$$q_t^W = (1-\beta)Bx_t + (1-\alpha)(1-\beta)\left(\tfrac{\alpha^{\alpha}A}{B}\right)^{\frac{1}{1-\alpha}}B \quad \text{and} \quad u_t^W = \left(\tfrac{B}{\alpha A}\right)^{\frac{1}{1-\alpha}} x_t. \tag{35}$$

The following difference equation describes the dynamics of the state-like variable x^W:[8]

$$x_{t+1}^W = \frac{\frac{(1-\alpha+\alpha\beta)B}{\alpha} x_t^W - (1-\alpha)(1-\beta)\left(\frac{\alpha^{\alpha}A}{B}\right)^{\frac{1}{1-\alpha}}B}{B - B\left(\frac{B}{\alpha A}\right)^{\frac{1}{1-\alpha}} x_t^W}. \tag{36}$$

[7] The expressions state-like and control-like variable stem from Mulligan and Sala-i-Martin [9]. In the following, we adopt their terminology.

[8] Compare the system of difference equations (28) and (29) above.

A search for the steady state of x^W yields the following quadratic equation:

$$0 = \left(x_{ss}^W\right)^2 + \tfrac{(1-\alpha)-\alpha(1-\beta)}{\alpha} \left(\tfrac{\alpha A}{B}\right)^{\frac{1}{1-\alpha}} x_{ss}^W - \tfrac{(1-\alpha)(1-\beta)}{\alpha} \left(\tfrac{\alpha A}{B}\right)^{\frac{2}{1-\alpha}} . \tag{37}$$

Solving for the two possible steady states $x_{ss_1}^W$ and $x_{ss_2}^W$ gives:

$$x_{ss_1}^W = (1-\beta) \left(\tfrac{\alpha A}{B}\right)^{\frac{1}{1-\alpha}} \qquad \text{and} \qquad x_{ss_2}^W = -\tfrac{1-\alpha}{\alpha} \left(\tfrac{\alpha A}{B}\right)^{\frac{1}{1-\alpha}} . \tag{38}$$

Since non-positive values for x make no sense in terms of the model considered, we henceforth focus on $x_{ss_1}^W = x_{ss}^W$ and ignore fixed point $x_{ss_2}^W$.

Second, we turn to the function V in (16). The control-like variable q_t^V implied by V is:

$$q_t^V = (1-\beta)^{1-\alpha} (1-\alpha\beta) A \left(x_t^V\right)^\alpha , \tag{39}$$

and the optimal human capital allocation u_t^V is given by $1-\beta$. Equation (34) then implies that the dynamics of x^V are determined by:

$$x_{t+1}^V = \frac{\alpha A}{B} (1-\beta)^{1-\alpha} \left(x_t^V\right)^\alpha . \tag{40}$$

Hence, the unique[9] steady state is positive and given by $x_{ss}^V = x_{ss}$. Moreover x_{ss} is the common fixed point of the functions V and W. A local analysis around this steady state reveals the system's saddle path property with stable eigenvalue α. Therefore we know that the phase plane diagram associated with the solutions to the Bellman equation (13) obeys exactly two trajectories that intersect in the steady state. Our candidates for the value function therefore correspond to the stable and to the unstable solution branches. Furthermore, we conclude that there are indeed no other solutions to the Bellman equation (13). Proposition 4 characterizes the stability properties of the fixed point x_{ss} in greater detail.

Proposition 4 (Properties of the saddle point) *The common fixed point x_{ss} of V and W is attracting for V and repelling for W.*[10]

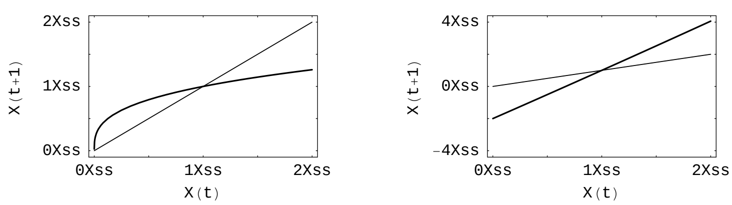

Figure 1: Dynamics of the state-like variable x

[9] As noted above, the trivial solution $k = x = 0$ is not considered here.

[10] For a formal definition of attractors and repellors in discrete dynamic systems, the reader is referred to Shone [11], Section 3.4.

The proof of Proposition 4 is given in the appendix. This result is illustrated graphically in the two diagrams of Figure 1, where tomorrow's state-like variable x_{t+1} is plotted as a function of today's state x_t. The left diagram displays the dynamics induced by the stable solution $V(k,h)$, while the right diagram refers to the unstable solution $W(k,h)$.[11] The function V generates the thick graph, that is flatter than the thin 45° line. No matter what initial value x_0 we choose in this case, the state-like variable will always converge to the steady state. The right diagram is rescaled, but the thin graph is still the 45° line. The thick line implied by using now the function W instead, however, is steeper than the 45° line.[12] As a consequence, the function W will lead to implausible results. Either the bounding conditions for the controls are violated or the state-like variable will diverge which leads to an implausible ratio of physical to human capital.

4.3. The phase diagrams for the Robinson Crusoe economy

In Figure 1 we have focused on the dynamics of the state-like variable x. The following Figure 2 in contrast contains the phase diagrams of the control-like variable q and of the control u.

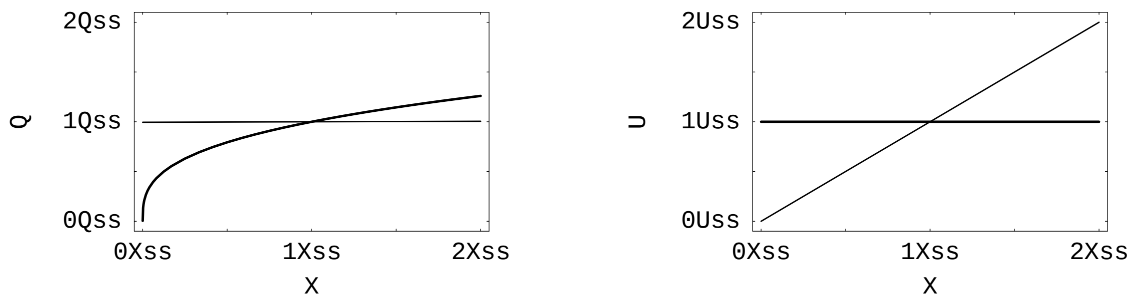

Figure 2: Phase diagrams for the control-like variable q and for the control u

The left panel shows the (x,q) space. The stable solution under the above calibration implies that the control-like variable q_t, which is plotted by the thick line in Figure 2, is a constant fraction of tomorrow's state x_{t+1} (cf. equations 39 and 40). This constant savings-consumption ratio[13] is due to the fact that Crusoe's logarithmic utility function gives rise to exactly offsetting income and substitution effects. The unstable solution, however, is represented by the slightly increasing thin graph with a relatively high intercept on the q axis. Hence, for low x the rescaled consumption q is too high and physical capital investment is too low and may even turn negative.

The right panel refers to the solutions in the (u,x) space. Again, the thick line corresponds to the real value function V, and the thin line displays the unstable solution

[11] The parameter calibration is as follows: $A = 1$, $B = 0.1$, $\alpha = \frac{1}{3}$, and $\beta = 0.99$.

[12] To be precise, the slope in the left panel is α and its counterpart in the right panel is $\frac{1}{\alpha\beta}$, i.e. the quotient of the eigenvalues associated with the system of equations (28) and (29). See also the appendix.

[13] Note that $q_t/x_{t+1} = B\beta c_t/k_{t+1}$ holds.

W. Note that the stable human capital allocation u is constant. This is again due to Crusoe's preferences because the opportunity costs of schooling are constant. The unstable value of u is starting in 0 and linearly increasing in x. Hence, this solution implies less schooling efforts when human capital is scarce and more schooling efforts when the endowment with human capital is high.

In the next section, we use the homogeneity in the initial conditions to introduce the 'value-function-like function'. We also document its connection to the value function.

5. The value-function-like function

Homogeneity in initial conditions was summarized in our observation (33). Note that for $\{c_t\}_{t=0}^{\infty}$ and $\{\tilde{c}_t\}_{t=0}^{\infty}$, this finding implies the following equality for the discounted lifetime utility streams:

$$\sum_{t=0}^{\infty} \beta^t \big(\ln c_t - \ln h_0\big) = \sum_{t=0}^{\infty} \beta^t \big(\ln \tilde{c}_t - \ln \tilde{h}_0\big). \tag{41}$$

Along the optimal decision paths, the first parts of both series converge and are nothing else than the value function given the respective initial capital stocks. The second parts also converge and we obtain the following equation:

$$V(k_0, h_0) - \tfrac{\ln h_0}{1-\beta} = V(\tilde{k}_0, \tilde{h}_0) - \tfrac{\ln \tilde{h}_0}{1-\beta}. \tag{42}$$

Without loss of generality, any solution $V(k,h)$ can therefore be deduced from $V(x,1) = f(x)$, where $f(\frac{k}{h}) = f(x)$ is referred to as the value-function-like function.[14] As a consequence, the model's homogeneity permits us to introduce f as follows:

$$V(k,h) = f(x) + \tfrac{\ln h}{1-\beta}. \tag{43}$$

For the Crusoe economy, one finds indeed the following functions f_V and f_W when bringing the two explicit solutions (16) and (17) in the above form:

$$f_V(x_t) = \theta + \theta_k \ln x_t \qquad \text{and} \qquad f_W(x_t) = \varrho + \varrho_1 \ln\left(x_t + \varrho_2\right). \tag{44}$$

Note that the shadow values of the two capital stocks, V_k and V_h, can be reformulated in terms of the derivative f' of the value-function-like function:

$$V_k(k,h) = \tfrac{1}{h} f'(x) \qquad \text{and} \qquad V_h(k,h) = \tfrac{1}{h}\big(\tfrac{1}{1-\beta} - x f'(x)\big). \tag{45}$$

Moreover, we can state the optimal controls (14) and (15) in terms of f':

$$u_t^* = \left(\tfrac{f'(x_{t+1})(1-\alpha)A}{B\left(\frac{1}{1-\beta} - x_{t+1} f'(x_{t+1})\right)}\right)^{\frac{1}{\alpha}} x_t \qquad \text{and} \qquad q_t^* = \tfrac{B(1-u_t^*)}{\beta f'(x_{t+1})}, \tag{46}$$

[14] The terminology is chosen in analogy to Mulligan and Sala-i-Martin [9].

The value-function-like function, therefore, provides an alternative approach when using the guess-and-verify method to determine an unknown closed-form solution. The first step in this strategy would be the substitution of the controls (46) into the following homogeneous version of the Bellman equation (13):

$$f(x_t) = \max_{q_t, u_t} \left\{ \ln(q_t) + \beta f(x_{t+1}) + \tfrac{\beta \ln B + \beta \ln(1-u_t)}{1-\beta} \right\} \quad \text{s.t. } (34), \tag{47}$$

Although the introduction of the value-function-like function reduces the dimensionality of the problem at hand, it does not guarantee to find an analytical solution. It does, however, pave the way to a numerical solution. The following section considers a continuous time framework to demonstrate the underlying idea.

6. Continuous time

6.1. A simple numerical method

By assuming that the two capital stocks do not depreciate at all, we can easily get another stylized model that is suitable for our switch to continuous time. If ρ denotes the subjective discount rate, the corresponding DOP can be obtained by replacing the sum in (5) with an integral and the discount factor β^t with $e^{-t\rho}$. Furthermore, k_{t+t} and h_{t+t} in (6) and (7) must be substituted by $\dot{k}_t$ and $\dot{h}_t$. In this case, the Hamilton-Jacobi-Bellman (differential) equation, the continuous time analogue to the discrete time Bellman (difference) equation (13), associated with Crusoe's DOP is given by:

$$\rho V = \max_{u,c} \left\{ \ln c + V_k (A k^\alpha u^{1-\alpha} h^{1-\alpha} - c) + V_h B(1-u)h \right\}. \tag{48}$$

On the left-hand side, the annuitized value of life-time utility, $V = V(k, h)$, appears. On the right-hand side, we have the current utility stream plus the current change in Crusoe's wealth position, which itself consists of two flows. The first flow is due to the change in the physical capital stock $\dot{k}_t$ and the second is due to the rate of change $\dot{h}_t$. Both flows are transformed in utility units by their respective shadow values. Note that the discussion on homogeneity in initial conditions and therefore on state-like and control-like variables as well as on value-function-like functions is still valid in the continuous time context:

$$x = \tfrac{k}{h}, \qquad q = \tfrac{c}{h}, \qquad \text{and} \qquad V(k, h) = f(x) + \tfrac{1}{\rho} \ln h. \tag{49}$$

Again we can express the shadow values in terms of the derivative of the value-function-like function, $f'(x)$:

$$V_k(k, h) = \tfrac{1}{h} f'(x) \qquad \text{and} \qquad V_h(k, h) = \tfrac{1}{h}\big(\tfrac{1}{\rho} - x f'(x)\big). \tag{50}$$

Note that these results are indeed very similar to those in (45). The homogeneous form of the Hamilton-Jacobi-Bellman (HJB) equation is given by:

$$\rho f(x) = \max_{u,q}\bigl\{\ln q + f'(x)(Ax^\alpha u^{1-\alpha} - q) + (\tfrac{1}{\rho} - xf'(x))B(1-u)\bigr\}. \tag{51}$$

As a consequence, we obtain the following optimal controls:

$$q^* = \tfrac{1}{f'(x)} \qquad \text{and} \qquad u^* = \Bigl(\tfrac{(1-\alpha)A}{B(\frac{1}{\rho f'(x)} - x)}\Bigr)^{\frac{1}{\alpha}} x. \tag{52}$$

We stress the difference to the discrete time framework. Insertion of the optimal controls (46) into (47) leads to a functional equation that is evaluated at two different points in time, where the function f and its derivative are even convoluted. With continuous time in contrast, insertion of the controls (52) into the homogeneous HJB equation (51) gives us a relatively simple implicit ordinary differential equation:[15]

$$\rho f(x) = \alpha\bigl(Af'(x)\bigr)^{\frac{1}{\alpha}}\Bigl(\tfrac{1-\alpha}{B(\frac{1}{\rho} - xf'(x))}\Bigr)^{\frac{1-\alpha}{\alpha}} x - \ln f'(x) + \tfrac{B-\rho}{\rho} - Bxf'(x). \tag{53}$$

Indeed, one can easily verify that a modification of the unstable solution (44) to the discrete time model solves the above differential equation:

$$f_W(x) = \tfrac{B + \rho\ln\rho - \rho}{\rho^2} + \tfrac{1}{\rho}\ln\Bigl(x + \tfrac{1-\alpha}{\alpha}\bigl(\tfrac{A\alpha}{B}\bigr)^{\frac{1}{1-\alpha}}\Bigr). \tag{54}$$

To examine its stability properties in the continuous time case we use (52) under $f_W(x)$ and look at the impied dynamics for the state-like variable x:

$$\dot{x}_t = B\bigl(\tfrac{B}{A\alpha}\bigr)^{\frac{1}{1-\alpha}} x_t^2 + \bigl(\tfrac{B(1-\alpha)}{\alpha} - \rho\bigr)x_t - \tfrac{\rho(1-\alpha)}{\alpha}\bigl(\tfrac{A\alpha}{B}\bigr)^{\frac{1}{1-\alpha}}. \tag{55}$$

On the positive axis, $\dot{x}_t$ only vanishes for the value

$$x_{ss} := \tfrac{\rho}{B}\bigl(\tfrac{A\alpha}{B}\bigr)^{\frac{1}{1-\alpha}} .... \tag{56}$$

Linearizing the right-hand side of equation (55) around the steady state x_{ss} shows that x_{ss} is locally unstable:

$$\tfrac{\partial \dot{x}_t(x_{ss})}{\partial x} = \tfrac{B(1-\alpha)+\rho\alpha}{\alpha} > 0. \tag{57}$$

We infer that (54) yields the unstable solution branch in the phase diagram. Finding an analytic expression for the true value function, however, seems to be a daunting task. Nevertheless, even without the additional knowledge of a solution to the HJB-equation, the introduction of the value-function-like function helps us to simplify the numerical analysis. Using f it is possible to transform the HJB equation into an explicit initial

[15] A differential equation of order n with form $G(x, f', .., f^{(n)}) = 0$ is called implicit, the form $G(x, f', .., f^{(n-1)}) = f^{(n)}$ is called explicit.

value problem for a one-dimensional ordinary differential equation, which can be solved by standard numerical schemes. Note that equation (53) has the form:

$$f(x) = G(x, f'(x)) \quad \text{for some function } G. \tag{58}$$

Up to an additive constant $(B/\rho - 1)$ and a factor (ρ), the function G equals the Hamiltonian of the transformed DOP. Denoting by G_p the derivative of function G with respect to f', this representation readily yields an explicit differential equation for the derivative of the value-function-like function f', namely: $f' = G_x(x, f') + G_p(x, f')f''$. Therefore, we arrive at the following explicit ordinary differential equation:

$$f''(x) = \tfrac{f'(x) - G_x(x, f'(x))}{G_p(x, f'(x))}, \tag{59}$$

which permits a simple analytical investigation of the value function as a parameter-depending differential equation. Together with some specific value of f, e.g. at the steady state, we are led to an explicit initial value problem, which can be solved by standard numerical schemes.

Because of saddle path stability, the unstable solution (54) as well as the true value function both solve the initial-value problem. This is due to the saddle path behavior at $(x_{ss}, f'(x_{ss}))$ in equation (59). To be precise, both numerator and denominator vanish in x_{ss} and the right-hand side is indeterminate. In order to obtain determinacy, we use L'Hôpital's rule:

$$f''(x_{ss}) = \tfrac{f''(x_{ss}) - G_{xx}(x_{ss}, f'(x_{ss})) - G_{xp}(x_{ss}, f'(x_{ss}))f''(x_{ss})}{G_{px}(x_{ss}, f'(x_{ss})) + G_{pp}(x_{ss}, f'(x_{ss}))f''(x_{ss})}. \tag{60}$$

Therefore, we consider a quadratic equation in $f''(x_{ss})$ that leads us to the respective initial values for analyzing numerically the initial value problems of the unstable solution and particularly of the stable solution. Hence, a local analysis around the steady state already provides enough information to apply numerical schemes. At this point the second derivative $f''_W(x)$ of our unstable solution in (54) simplifies determination of its stable counterpart:

$$f''(x_{ss}) = \tfrac{G_{xx}(x_{ss}, f'(x_{ss}))}{f''_W(x_{ss})G_{pp}(x_{ss}, f'(x_{ss}))}. \tag{61}$$

In this sense, our additional knowledge of $f_W(x)$ can facilitate the calculations in the saddle-point case.

When asking the computer to solve the differential equation (59), indeterminacy in the steady state requires an initial value outside steady state. If $\varepsilon > 0$ denotes a sufficiently small real number, the concrete initial value used in the numerical implementation may be the following:

$$f'(x_{ss} \pm \varepsilon) = f'(x_{ss}) \pm \varepsilon f''(x_{ss}). \tag{62}$$

Note that via (46) the resulting initial value problem can also be stated in terms of the control-like variable q or in terms of u instead of using f'.

6.2. The role of logarithmic preferences

At first glance, it seems that we always obtain the desired representation (58). However, the reduction to state-like and control-like variables may introduce a non-derivative term on the right. For instance, the use of an isoelastic utility function $g(c) = c^{1-\sigma}$, where $\sigma \in (0, 1)$ is the inverse of the intertemporal elasticity of substitution, lets us set by homogeneity:

$$V(k, h) = f(x)h^{1-\sigma}. \tag{63}$$

In addition to the derivative of the value-function-like function, the shadow value of human capital now also includes a non-derivative term:

$$V_k = h^{-\sigma} f'(x) \qquad \text{and} \qquad V_h = h^{-\sigma}\big((1-\sigma)f(x) - xf'(x)\big), \tag{64}$$

such that the optimal controls become more complex in this case:

$$u^* = \left(\tfrac{(1-\alpha)Af'(x)}{B((1-\sigma)f(x)-xf'(x))}\right)^{\frac{1}{\alpha}} x \qquad \text{and} \qquad q^* = \left(\tfrac{1-\sigma}{f'(x)}\right)^{\frac{1}{\sigma}}. \tag{65}$$

Insertion of these controls into the corresponding homogeneous HJB equation reveals the consequences:

$$\rho f = \sigma\big(\tfrac{1-\sigma}{f'}\big)^{\frac{1-\sigma}{\sigma}} + \alpha\big(Af'\big)^{\frac{1}{\alpha}}\big(\tfrac{1-\alpha}{B((1-\sigma)f-xf')}\big)^{\frac{1-\alpha}{\alpha}} x + B\big((1-\sigma)f - xf'\big). \tag{66}$$

A separation of f like in (58) is impossible and we conclude that it is the additive structure in (43) that allows the simple solution method advocated in Section 6.1. The appearance of f in a shadow value, which is due to the more general isoelastic preferences, drastically complicates the structure of the reduced homogeneous HJB equation and more sophisticated methods are needed for its analysis.

7. Conclusion

In economic modeling, saddle path stability and homogeneity in initial conditions can be found over and over again. By using the simple discrete time Robinson Crusoe model, this paper illustrates both concepts in detail. Since we solve the model explicitly, we forgo complicated mathematical analysis and focus instead on the economic meaning. Saddle path stability requires two solutions to Bellman's functional equation of the respective optimization problem. Moreover, both functions must intersect in the saddle point. We show that the saddle point is indeed attracting the model's state when using the value function. For the second solution in contrast the saddle point is a repelling fixed point. Despite their important and intuitive economic meaning, transversality conditions are sometimes omitted in theoretical work. In the context of our model, the transversality conditions enable us to discriminate between the two solutions to the Bellman equation and therefore tell us which of them is the true value function.

Homogeneity in the initial conditions is a property that allows the introduction of state-like and control-like variables. Thereby it reduces the dimensionality of the problem. With respect to the functional equation, it allows the introduction of the value-function-like function. A step that drastically simplifies analysis in the continuous time case, since it permits us to transform the optimization problem into an initial value problem for one ordinary differential equation. Standard and easy to handle numerical schemes can therefore be used for its analysis. Applicability of this method, however, is directly linked to Crusoe's preferences. Logarithmic preferences, i.e. the case when the income and substitution effects exactly offset, are a necessary prerequisite for its use. For in this case, the resulting structure of the HJB equation is relatively simple.

Appendix

A.1 Proof of Proposition 1

Insertion of the function V into the Bellman equation and determining the second derivatives with respect to the controls gives us the following Hessian matrix:

$$H = \begin{bmatrix} \frac{-1}{\alpha\beta(1-\alpha\beta)^2 y_t^2} & \frac{1-\alpha}{\alpha\beta(1-\alpha\beta)(1-\beta)y_t} \\ \frac{1-\alpha}{\alpha\beta(1-\alpha\beta)(1-\beta)y_t} & \frac{-(1-\alpha)(1-\alpha\beta(1-\alpha))}{\alpha\beta(1-\alpha\beta)(1-\beta)^2} \end{bmatrix},$$

where the controls (19) have been used. The upper left entry of this matrix is negative and the determinant of H is positive:

$$det\,(H) = \frac{1-\alpha}{\alpha\beta^2\,(1-\alpha\beta)^3\,(1-\beta)^2\,y_t^2} > 0.$$

Hence, the quadratic form is negative definite and the controls are indeed necessary and sufficient for a maximum.

Insertion of the function W into the Bellman equation and determining the second derivatives with respect to the controls gives us the following Hessian matrix:

$$H = \frac{1}{1-\beta} \begin{bmatrix} \frac{-1}{(B\beta k_t+(1-\alpha)\beta(\frac{\alpha^\alpha A}{B})^{\frac{1}{1-\alpha}}h_t)^2} & 0 \\ 0 & \frac{-\alpha(1-\alpha)y_t}{(B\beta k_t+(1-\alpha)\beta(\frac{\alpha^\alpha A}{B})^{\frac{1}{1-\alpha}}h_t)u_t^2} \end{bmatrix},$$

For sensible values of the capital stocks, i.e. $k_t > 0$ and $h_t > 0$, the upper left entry of this matrix is negative and the determinant of H is positive. Hence, the quadratic form is negative definite and the controls are indeed necessary and sufficient for a maximum.$\square$

A.2 Proof of Proposition 3

First note, that under W the transversality conditions (25) become

$$\lim_{T\to\infty}\frac{\beta^T}{c_T}Bk_T=0 \qquad \text{and} \qquad \lim_{T\to\infty}\frac{\beta^T}{c_T}\frac{1-\alpha}{\alpha}\left(\frac{\alpha A}{B^\alpha}\right)^{\frac{1}{1-\alpha}}h_T=0.$$

Using $c_T=(\beta B)^T c_0$ and Proposition 2 for k_T, the first transversality condition reads as follows:

$$\lim_{T\to\infty}\frac{\beta^T Bk_T}{c_T}=\frac{(1-\alpha)\,B}{(1-\alpha\beta)\,(1-\beta)\,c_0}\lim_{T\to\infty}\frac{k_0-x_{ss}h_0}{\alpha^T}.$$

Thus, there are three cases possible:

$$\lim_{T\to\infty}\frac{\beta^T Bk_T}{c_T}=\begin{cases}+\infty & \text{if } \frac{k_0}{h_0}>x_{ss},\\ 0 & \text{if } \frac{k_0}{h_0}=x_{ss},\\ -\infty & \text{if } \frac{k_0}{h_0}<x_{ss}.\end{cases}$$

For the second transversality condition we obtain:

$$\lim_{T\to\infty}\frac{1-\alpha}{B^T c_0}\left(\frac{\alpha A}{B^\alpha}\right)^{\frac{1}{1-\alpha}}\frac{h_T}{\alpha}=\frac{(1-\alpha)\,B}{(1-\alpha\beta)\,c_0}\lim_{T\to\infty}\frac{x_{ss}h_0-k_0}{\alpha^T}.$$

Again one can distinguish between three cases:

$$\lim_{T\to\infty}\frac{\beta^T}{c_T}\frac{(1-\alpha)\,B}{\alpha}\left(\frac{\alpha A}{B}\right)^{\frac{1}{1-\alpha}}h_T=\begin{cases}-\infty & \text{if } \frac{k_0}{h_0}>x_{ss},\\ 0 & \text{if } \frac{k_0}{h_0}=x_{ss},\\ +\infty & \text{if } \frac{k_0}{h_0}<x_{ss}.\end{cases}$$

Theorem 4.15 in Stokey and Lucas [12] shows that the sequences of the state variables k_t and h_t are a solution to Robinson Crusoe's DOP if and only if they satisfy the Euler equations and the transversality conditions. Hence we learn that V always gives us the utility maximizing controls while W in general leads us to wrong results. Just if $\frac{k_0}{h_0}=x_{ss}$ holds, the function W solves Crusoe's problem. Therefore, V is the true value function.□

A.3 Proof of Proposition 4

First, we consider V. We will argue that whatever initial state $x_0:=Mx_{ss}$, with $M\in\mathbb{R}_{++}$, we choose the function V implies $x_T=M^{\alpha^T}x_{ss}$ such that the model finally converges to x_{ss}. The proof is by induction. Let $T=1$ then starting with $\frac{k_t}{h_t}=Mx_{ss}$ where $M\in\mathbb{R}_{++}$ gives:

$$x_{t+1}=(1-\beta)^{1-\alpha}\frac{\alpha A}{B}\left(Mx_{ss}\right)^\alpha=M^{\alpha^1}x_{ss}$$

$T \mapsto T+1$: Suppose $x_{t+T} = M^{\alpha^T} x_{ss}$ holds. Then

$$x_{t+T+1} = (1-\beta)^{1-\alpha} \frac{\alpha A}{B} \left(M^{\alpha^T} x_{ss}\right)^{\alpha} = M^{\alpha^{T+1}} x_{ss}$$

Hence, no matter how we choose $M \in R_{++}$, i.e. where x_t starts on the positive axis, the state-like variable x will finally converge to x_{ss}

$$\lim_{T\to\infty} x_{t+T} = \lim_{T\to\infty} M^{\alpha^T} x_{ss} = x_{ss}$$

and x_{ss} is an attracting fixed point of V.

Next, we consider W. The proof exploits two facts. First, the function of the state dynamics $x^W_{t+1}\left(x^W_t\right)$ is continuously differentiable in the neighborhood of x_{ss}. Second, if the slope of this derivative is bigger than one in absolute value, the fixed point is repelling. We obtain:

$$\frac{\partial x^W_{t+1}\left(x_{ss}\right)}{\partial x^W_t} = \frac{\frac{1-\alpha+\alpha\beta}{\alpha} - \frac{(1-\alpha)(1-\beta)}{\alpha}}{\left(1-\left(\frac{B}{\alpha A}\right)^{\frac{1}{1-\alpha}} x_{ss}\right)^2} = \frac{1}{\alpha\beta} > 1.$$

Hence, x_{ss} is a repelling fixed point of W. □

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