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Abstract

This paper studies an environment in which a seller seeks to sell two different items to buyers. The seller designs a membership mechanism that assigns positive allocations to members only. A membership mechanism specifies a member set and a membership fee as well as allocation and payment rules for the two goods. Among such mechanisms with bundling participation, we first establish the revenue-maximizing allocation rule given a regularity condition for a modified valuation distribution that reflects the set, which resolves the existence problem of a member set and results in a simple optimal payment rule. The optimal allocation rule enables us to compare between membership and separate participation, suggesting conditions under which membership dominates separate participation: interplay between the number of bidders and the degree of the stochastic dominance of valuation distributions. This allocation also provides a rationale for secret reserve prices, a long-standing puzzle in theory.

Keywords and Phrases: mechanism design, membership, multidimensional types, auction, secret reserve prices

JEL Classification Numbers: D44, D82

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1 Introduction

Membership is ubiquitous. One common usage of group formation is membership for *sale*. Sale membership *bundles participation* to allow only its members to participate in multiple-item sale, in return for membership fees. Despite this widespread daily business practice, there is surprisingly little academic research on the dominance of bundling participation. That is, why should a seller prefer membership to *separate participation*?

The most familiar membership offers discounted repeated consumption to each single buyer over multiple periods, bundling "present participation and future participation" for identical items. This type of membership for screening a single buyer is a simple direct application of the well-established bundling goods; thus, there appears to be no need for separate research on it, at least from a rational point of view.[1] However, little is known about membership for *mechanism design* given multiple buyers. In the real world, such membership may not exist as a day-to-day form, but it has been popularly institutionalized in wholesale business and auction houses.[2] More importantly, lack of understanding of its "mechanism" may leave it under-utilized.[3] On the other hand, research on multidimensional screening has a tendency to exclusively focus on addressing bundling goods as if those were the only area to study the topic – in addition, it is impossible to obtain the full characterization for a mechanism with bundling goods.[4] Yet, study on multidimensional mechanism design is by no means limited to bundling goods, especially given those applications, *i.e.,* indirect mechanisms in which a seller bundles participation through a membership but *does not* bundle goods.[5] A mechanism for membership, therefore, has enormous potential for development;

[1]That is, in the monopoly screening case with a single agent, a mixed bundling for two goods can be converted into a membership, and vice versa. A seller's gain for multi-item *over time* – an item in the present period and the same item in a future period – like a gym membership toward a single buyer, especially as opposed to an institutional buyer, is more suitably approached with its behavioral aspect, in which a seller can exploit each buyer's overconfidence about future control, as, for example, interestingly observed by DellaVigna and Malmendier (2006).

[2]In particular, membership with premium membership fees is typical in the wholesale car sale and auctions. See recent empirical works, Lacetera et al (2016) and Larsen (2020) for the institutional details of the latter case, though their focus is not on membership.

[3]For example, currently, for spectrum auctions in the U.S., an upfront payment is required to guarantee sincere bidding by qualified bidders, but the payment is not a membership fee but refundable once an auction ends (see, B. Upfront payment of FCC-94-61A1).

[4]The setting with buyers having multidimensional valuations for multiple items being natural, the analytical difficulty and anomaly of it are well documented, even for mild extensions, such as a setting with only two items and a single agent. There is no full characterization for the optimal allocation, though the local dominance of bundling over separate sale is established by McAfee, McMillan and Whinston (1989) for mixed bundling, posting of both individual prices and bundle prices. The dominance of membership in the present paper is based on a different approach, and, further, is presented in the global sense, only with pure membership, *i.e.*, even without requiring ex-ante reserve prices.

[5]For instance, bundling participation typically comes with no bundling goods; that is, for members, those

the rationale behind it is yet to be explored.

This paper suggests a novel direction to study multidimensional screening with multiple buyers: bundling participation. A direct mechanism of bundling participation is called a *membership mechanism*. We study a natural class of indirect mechanisms with bundling participation, and thus a direct mechanism simply corresponds to an indirect mechanism from such a set in which each buyer can – only after becoming a member – purchase two items separately to reflect the actual practices. As a result, a membership mechanism bundles participation but requires separable incentive compatibility conditions for two goods given the separate purchase *if a buyer is a member*.[6] Yet, the decision to become a member depends on a multidimensional type from bundling participation – this is what makes the membership mechanism distinct from other multidimensional screening problems. Even in the absence of bundling goods, however, finding the optimal membership mechanism is a delicate problem since allocation and payment rules are entangled with a member set through the lowest type to participate.[7]

Under the domain, we obtain the revenue-maximizing membership mechanism. The first main result shows that among members, the buyer with the highest valuation for each item wins the item, if a *modified virtual valuation* distribution reflecting the restrictive member set satisfies monotonicity. That is, for the optimal allocation, the seller assigns each good as if he "ignores" a member set. Hence, the finding resembles the single dimension's optimal allocation. However, to do that for a membership mechanism with two goods, the seller has to have more *control* of the mechanism to the extent that a member's allocation for one good depends on valuations of the other good and it also depends on a non-member's valuations. Additionally, we need to solve the problem without being able to pin down the payoff of a buyer with the lowest valuation in each dimension, unlike the case in a single dimension; in multi-dimensions, the buyer's payoff and his or her decision to become a member depend on valuation for the other good as well. Essentially, the characterization greatly simplifies the problem of choosing a member set and a membership fee to merely choosing a membership fee.

The full characterization enables us to compare membership with separate participation. Since bundling participation is equivalent to *bundling exclusion*, the comparison can be rein-

monopolistic auction houses auction for good A and good B separately, with no auctioning for a bundle of the two goods.

[6]Separate purchase in an indirect mechanism requires separable incentive compatibility conditions given two goods for members, which can be shown by the "extended" revelation principle in Section 3. In addition, we use the term "separate purchase" differently from the term "separate participation" in the present paper; in short, the former is equivalent to no bundling of goods, and the latter is separate exclusion.

[7]This is also related to an existence problem. That is, in nature, the existence of a member set satisfying such incentives to become a member involves a *fixed point* type argument. Specifically, a membership mechanism assigns positive allocations to members only. As a result, the mechanism's allocation and payment may depend on a member set that the seller adopts, which in turn generates each buyer's payoff, determining his or her benefit of becoming a member; or a member set.

terpreted as that between bundling exclusion and separate exclusion. In a single dimension, the exclusion of low valuations in the optimal mechanism is well known by Myerson (1981). In two dimensions, the key idea for the dominance of membership is *not* to exclude all of the low valuations for each item; instead, it is to exclude only those two-dimensional buyer types that have low valuations for the other item as well. In other words, the membership includes a buyer type with a low valuation for a given good if it has a relatively high valuation in the other dimension, and by doing so, aims to attract more participating buyer types and, therefore, collect more membership fees.

Yet, one is naturally led to wonder if this way of attracting greater participation *contradicts* the very nature of the optimal mechanism in a single dimension; that is, solely from the perspective of the sale of good A, what is the benefit of having the possibility that *low* winning bids for good A realize, by allowing those bidders to participate just because they have high valuations for the other good, good B? We show that such a "sacrifice" in the bid is bounded from below if the number of bidders is sufficiently large. Then, in return for the bounded loss, the seller can collect membership fees from a larger set of types that participate; the benefit outweighs the loss for a sufficiently large number of bidders.

Although it appears that the dominance result is intuitive, finding a closed-form solution for the membership mechanism is intractable in general, which is a separate challenge for the comparison with separate participation or exclusion. To tackle it, we find a link between the two *different mechanisms*, membership and separate participation, by connecting a member set *frontier*'s intercept to an optimal reserve price from separate participation.[8] In membership, as already mentioned – unlike separate exclusion – a bidder whose valuation is low in one dimension can be a member if the other valuation is sufficiently high. Hence, those low bidder types' expected payments become an additional source for the seller's revenue but the realized lower winning bids can negatively affect him. The *net effect* is always negative on account of the same comparison from the optimal exclusion by Myerson (1981). However, with the connection, we establish that the negative-net-effect's lower bound is *dominated* by the membership fees collected from three categories of bidder types – a (relatively) high valuation in each dimension with a low valuation in the other and high valuations for both dimensions – if the number of bidders is large.

The opposite case is also examined. The optimal separate participation dominates the optimal membership mechanism if bidders' valuation distributions become sufficiently stochastic dominant. Under the condition, in membership, the revenue decrease from low bids dominates the additional revenue from the expanded participation when valuation distributions include more high valuations.

This paper provides three extensions of the main model. The implementation of the optimal mechanism provides yet another interesting result, with respect to its connection to secret reserve prices. In a mechanism, even a non-member "reports" his or her type, but

[8] A direct approach that compares closed-form solutions shown in Section 5 for an illustrative example with a uniform distribution.

in an actual auction, a non-member simply does not bid; implementing the simple optimal mechanism may not be simple. Yet, its auctioning is achievable with secret reserve prices. The optimal membership mechanism of this paper provides a rationale for it. The setting in which a seller has two items can also be applied to a monopolistic *auction house or platform* with two different sellers paying a fixed rate of a final price as each seller's fee for the platform. In addition to the two aforementioned extensions, the main results written for "pure membership," *i.e.*, membership with *no* ex-ante reserve price, is also generalized in a section.

Primarily, we aim to suggest a new and different direction, bundling participation, in the multidimensional screening literature with multiple agents.[9] The participation or exclusion in a single dimension is a well-known key feature of the characterization by Myerson (1981). In auctions, the optimal reserve price can be derived from the trade-off between the set of types that are excluded and the minimum bid for a winner. The problem is also rigorously treated by Stegeman (1996) for entry with costs. The one-dimensional problem makes it unnecessary to assume any property for a "member set," unlike a member set in this multi-dimension case. The present paper's setting with incomplete information and two items can facilitate the comparison between its performance and that of its counterpart, separate participation by Myerson (1981). To clarify the present paper's different approach, therefore, a new rationale for the dominance of membership is thoroughly studied.

The robustness of a non-negligible set of not-participating buyer types in multi dimensions was well established by Armstrong (1996) and Rochet and Choné (1998), but the full characterization of the optimal allocation for participating buyer types is intractable.[10] In a restricted domain, we provide the full characterization of membership mechanism. All in all, the global approach not only handles the existence problem but also enables us to identify conditions under which membership dominates separate participation and the opposite case as well.

The prevalence of secret reserve prices has long been an important puzzle, especially as it marks the discrepancy between theory and empirical findings, which is deftly explained

[9]Membership with *a single buyer* as opposed to membership mechanism with multiple buyers, as articulated, has no bite due to its close proximity to bundling goods. That is, by basic linear algebra, for each mixed bundling with p_A, p_B and p_{AB}, we can have a membership with p'_A, p'_B, e, where e is a membership fee that satisfies $p'_A + e = p_A$, $p'_B + e = p_B$ and $p'_A + p'_B + e = p_{AB}$ simultaneously and vice versa. Hence, the focus of this paper, a membership mechanism with *multiple* buyers, differs from the literature on bundling goods (see Manelli and Vincent (2006) and Manelli and Vincent (2007) among others).

[10]The anomaly of multidimensional screening was reported by the successive contributions: Thanassoulis (2004) and Manelli and Vincent (2006), and further by Hart and Reny (2015) and Hart and Nisan (2017). The complexities are not our main concern in this model with the buyer's separate purchase. The study on this restricted domain enables us to find the optimal allocation for members of this model so that we can compare its performance with separate participation. Yet, it is critical for the seller of this model to utilize a modified *joint* distribution that incorporates a member set, despite the independence of two valuations, which is different from Carroll (2017) who considers a robust principal problem such that he has beliefs only about each marginal distribution.

by Jehiel and Lamy (2015) in their behavioral approach.[11] However, there is no satisfactory rational view for the most standard case – independent and private values – to resolve the puzzle. The optimal membership mechanism of this paper provides a rationale for it.

In addition, the present paper is related to two more strands of literature. The multi-object aspect has been not investigated in the two-sided market literature (see, Caillaud and Jullien (2003), Rochet and Tirole (2003, 2006) and Armstrong (2006)).[12] The simple characterization of this model can make such a setting apply to the literature. Since a membership fee constrains a buyer's feasible choices, the seller's optimal membership fee is derived from the trade-off between them. That is, the fee may constrain the extent to which buyers can pay for the two goods, so in that sense, it is related to auctions with budget constraints (see Che and Gale (1998, 2000), Pai and Vohra (2014) and Benoit and Krishna (2001)).[13]

The model is in Section 2, and the membership mechanism is introduced in Section 3 with an illustrative example. Sections 4 and 5 provide the main results: the optimal characterization of membership and its dominance over separate participation. The dominance of separate participation is found in Section 6, and the extensions are in Section 7. The concluding remarks are in the last section. All the proofs are collected in an appendix.

2 Model

One seller seeks to sell two non-identical items to $N \geq 2$ potential buyers. Each of the two items, good A and good B, is a single unit of an indivisible good. Buyer $i \in I \equiv \{1, ..., N\}$ receives value v_i from good A and value w_i from good B. We suppose that the two goods' valuations are not related so that the dominance holds even in the absence of correlation between them; for each buyer i, v_i and w_i are independently drawn from $[\underline{v}, \overline{v}]$ with $\overline{v} > \underline{v} \geq 0$,

[11] See Ashenfelter (1989) for art auctions and Bajari and Hortacsu (2003) for the use of secret reserve prices in online auctions. In addition, the aforementioned wholesale car auctions studied by Lacetera et al (2016) and Larsen (2020) typically institutionalize secret reserve prices. There is also a series of econometric studies for timber auctions with secret reserve prices (see Li and Perrigne (2003) among others). A secret reserve price in the present paper is secret, precisely, in the sense that it is random; the two concepts are used in a mixed way in the pertinent literature.

[12] The same industry could have two different business models, for example, *Coparts* for both direct sale and a platform in the wholesale car business, and *Manheim Auctions*, the largest wholesale auto auction, more for a platform. For interpretation of the model as an auction house or platform, see Section 7, where to focus on the multi-unit auction, we abstract from two-sided market pricing. Some auction houses or platforms require a fixed flat fee from a seller (with an additional fee above a certain amount), but others, like *eBay* or *Manheim Auctions*, require a fixed *percentage* of a final price from a seller. The question of how the seller's fee for auction houses can be chosen in multi-item auctions is beyond the scope of this paper.

[13] Of course, there is an intrinsic difference in deriving a seller's revenue between the model with a budget constraint or an entry cost and this model with a membership fee since a fee is a part of the revenue. See also nonidentical objects section in Krishna (2002) for auctions with budget constraints.

and $[\underline{w}, \overline{w}]$ with $\overline{w} > \underline{w} \geq 0$, respectively. In addition, for each good, valuations *across* buyers are independently and identically drawn according to a differentiable cumulative distribution function (CDF) F_k with density $f_k > 0$ for $k \in \{A, B\}$. A pair of buyer i's valuations (v_i, w_i) is called buyer i's type, and we denote it by $\theta_i \equiv (v_i, w_i) \in \Theta_i \equiv [\underline{v}, \overline{v}] \times [\underline{w}, \overline{w}]$ and a type profile by $\theta \equiv (\theta_1, ..., \theta_N) \in \Theta \equiv \Theta_1 \times \cdots \times \Theta_N$, where Θ_i denotes buyer i's type set; Θ a type profile set.

An outcome $x \equiv (x_A, x_B)$ specifies which good is assigned to buyer i or not sold, with a set of outcomes $X \equiv \{0\} \cup I \times \{0\} \cup I$. If $x_k = i$ for $i \in I$, good k is sold to buyer i, and if $x_k = 0$, it remains with the seller.[14] The probability that outcome x occurs is denoted by $q(x)$ and a set of all probability distributions over a set of outcomes X by $\Delta(X) \equiv \{\big(q(x)\big)_{x \in X} : q(x) \in \mathbb{R}_+ \text{ and } \sum_{x \in X} q(x) = 1\}$. The joint probability generates its marginal probabilities; for $x_A, x_B \in I$, $q_A(x_A) \equiv \sum_{x_B \in \{0\} \cup I} q(x)$ denotes the probability that good A is sold to buyer x_A, and $q_B(x_B) \equiv \sum_{x_A \in \{0\} \cup I} q(x)$ denotes the probability that good B is sold to buyer x_B. We assume that buyers are risk-neutral and that each buyer's total benefit from trade is the sum of the two valuations, so buyer i obtains expected payoff $q_A(i)v_i + q_B(i)w_i - t_i$ if he purchases two goods with probabilities $q_A(i)$ and $q_B(i)$, by paying a transfer t_i to the seller. The seller obtains the sum of transfers $\sum_{i \in I} t_i$ if for each $i \in I$, buyer i transfers t_i to him.

For each $k \in \{A, B\}$, it is assumed that F_k satisfies the standard monotone hazard rate condition such that $\frac{f_k(x)}{1 - F_k(x)}$ is nondecreasing, and that the type distribution F_k is common knowledge among the players. Finally, each buyer's reservation payoff is normalized to zero.

3 Membership mechanism

A full characterization of a multidimensional mechanism, even for a single buyer case, is not possible, although a simple description of it can be readily provided.[15] In this model, we consider a *natural* class of indirect mechanisms with membership, and characterize the optimal mechanism, in the global sense, among direct mechanisms that simply correspond to them.

3.1 Definition

The indirect mechanisms studied in this paper are selling mechanisms with membership. A buyer can be a member if he meets a certain member criterion and pays a fee, and only members can purchase either good. That is, the main subject of this paper concerns indirect mechanisms with *bundling participation*, not bundling goods. Specifically, in an indirect

[14] For example, $x = (0, 2)$ indicates that good A remains with the seller and good B is sold to buyer 2.

[15] A full direct mechanism for sale of two items can be defined by functions $(q, t_1, ..., t_N)$, where $q : \Theta \to \Delta(X)$ and $t_i : \Theta \to \mathbb{R}$, and if buyer i reports θ_i for all $i \in I$, the seller commits to an outcome x with probability $q(x|\theta)$ by collecting a transfer $t_i(\theta)$ from buyer i.

mechanism, there are two separate sale procedures (*e.g.,* auctions) *for members* and so two Bayesian Nash equilibria, $\boldsymbol{\beta}_A \equiv (\beta_A^1, ..., \beta_A^N)$ and $\boldsymbol{\beta}_B \equiv (\beta_B^1, ..., \beta_B^N)$, where $\beta_A^i : \Theta_i \to \mathbb{R}_+$ denotes member i's (bidding) strategy for good A and $\beta_B^i : \Theta_i \to \mathbb{R}_+$ denotes his strategy for good B. Despite the *separate purchase*, a decision to become a member depends on a multidimensional type from bundling participation, which is entangled with allocation and payment rules through the lowest type to participate. Thus, the characterization is still a delicate problem, requiring tractable but careful treatment. This model on multidimensional screening for bundling participation can also be reinterpreted as bundling exclusion, especially, in contrast to separate exclusion in Myerson (1981).

A direct mechanism of an indirect mechanism with membership is called a *membership mechanism* or **M**-mechanism. A membership mechanism bundles participation to allow only members to purchase two goods separately. In a direct mechanism, buyer i becomes a member if his reported type satisfies a member criterion $m(\theta_i) \geq 0$ and pays a membership fee $e \in \mathbb{R}$, where the function m is assumed to be continuous and weakly monotonic, including, for example, $m(\theta_i) = \max\{v_i, w_i\} - \frac{1}{2}$ in Figure 1.[16] A set of types satisfying the criterion, a set of member types, is denoted by $M(m)$ such that

$$M(m) \equiv \{\theta_i \in \Theta_i : m(\theta_i) \geq 0\}. \tag{1}$$

In addition, denote by $\overline{M}(m) \equiv \{\theta_i \in \Theta_i : m(\theta_i) = 0\}$ a set of member types that *minimally* satisfy the criterion. The two properties on the mapping $m(\cdot)$ make a member (type) set closed and connected, like a weakly preferred set from a standard utility theory.[17] We henceforth call the minimal member criterion a *member standard line* since the monotonicity also implies that $\overline{M}(m)$ is not "thick." With a member set $M(m)$, or simply M, an allocation rule for an **M**-mechanism is denoted by $q^M(x|\theta)$ and its marginal probabilities $q_A^M(x_A|\theta)$ and $q_B^M(x_B|\theta)$ can be derived accordingly, similar to $q_A(x_A)$ and $q_B(x_B)$ in the model section. With membership, the transfer t_i of buyer i can be divided into two parts, a membership fee e and a payment rule *only* for purchase of two goods. The separate purchase of the two goods requires two different transfers for them in a mechanism $t_A^M(i|\theta)$ and $t_B^M(i|\theta)$ such that for each $k \in \{A, B\}$, $t_k^M : \Theta \to \mathbb{R}^N$, where $t_k^M(i|\theta)$ is a transfer from buyer i for good k. Then, for θ in *ex post*, the seller obtains a sum of transfers $\sum_{i \in I^M(\theta)} [t_A^M(i|\theta) + t_B^M(i|\theta)] + \#I^M(\theta)e$, where a set of members is defined as $I^M(\theta) \equiv \{i \in I : \theta_i \in M(m)\}$.

Now, we define an **M**-mechanism formally.[18]

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[16] Formally, m satisfies the weak monotonicity if $(v_i', w_i') >> (v_i, w_i)$ implies $m(v_i', w_i') > m(v_i, w_i)$. Note for two vectors, a and b, we say that $a >> b$ if $a_k > b_k$ for all k; $a > b$ if $a_k \geq b_k$ for all k and $a \neq b$; and $a \geq b$ if $a_k \geq b_k$ for all k.

[17] As well-known from the classical utility theory, the continuity of m makes the upper contour set closed, and the monotonicity makes it connected.

[18] Alternatively, define a set of "effective" outcomes as $X^M(\theta) \equiv \{0\} \cup I^M(\theta) \times \{0\} \cup I^M(\theta)$. Then, a mechanism is said to be an **M**-mechanism if for all $i \in I$, there exists a continuous and monotonic function $m : \Theta_i \to \mathbb{R}$ such that for each $\theta \in \Theta$, the sum of an **M**-mechanism's probability allocations $q^M(x|\theta)$ for the effective outcomes is 1, *i.e.*, $\sum_{x \in X^M(\theta)} q^M(x|\theta) = 1$.

Definition 1 *A direct mechanism* $\Gamma = (q^M, t^M, m, e)$ *is an* $\boldsymbol{M}$*-mechanism if there exists a continuous and monotonic function* $m : \Theta_i \to \mathbb{R}$ *for all* $i \in I$ *such that for each* $\theta \in \Theta$*,*

$$q_A^M(i|\theta) = q_B^M(i|\theta) = 0 \text{ for all } \theta_i \in \Theta_i \setminus M(m). \tag{2}$$

The seller can assign a zero allocation of either good to a member but *always* assigns a zero allocation to a non-member. No condition for a non-member's payment is assumed, though zero transfer for a non-member is intuitive; it will be the outcome of the seller's revenue maximization. Note that the same member criterion $m(\cdot)$ applies to all i, and that the definition itself does not embody any incentive compatibility; that is, an **M**-mechanism is not necessarily incentive compatible and thus, to apply the revelation principle, we need to show it.

An alternative formal account of an **M**-mechanism can be provided as follows, using the language of a semidirect mechanism by Stegeman (1996). The principal designs a mechanism such that in a pre-stage, each buyer decides whether to enter or not, and *only* for those who enter and report their valuations, the seller commits to transferring item A to buyer i with probability $p_A(i|\theta)$, and each buyer i pays $x_A(i|\theta)$ and transferring item B to buyer i with probability $p_B(i|\theta)$, and each buyer i pays $x_B(i|\theta)$. Then, with them, $q_A^M(i|\cdot) : \Theta^n \to \Delta^n$, $t_A^M(i|\cdot) : \Theta^n \to \mathbb{R}$, $q_B^M(i|\cdot) : \Theta^n \to \Delta^n$ and $t_B^M(i|\cdot) : \Theta^n \to \mathbb{R}$ can be formalized such that

$$q_A^M(i|\theta) \equiv \begin{cases} p_A(i|\theta) & \text{if } \theta_i \in M(m), \\ 0 & \text{if } \theta_i \notin M(m), \end{cases} \quad t_A^M(i|\theta) \equiv \begin{cases} x_A(i|\theta) & \text{if } \theta_i \in M(m), \\ 0 & \text{if } \theta_i \notin M(m), \end{cases} \tag{3}$$

and

$$q_B^M(i|\theta) \equiv \begin{cases} p_B(i|\theta) & \text{if } \theta_i \in M(m), \\ 0 & \text{if } \theta_i \notin M(m), \end{cases} \quad t_B^M(i|\theta) \equiv \begin{cases} x_B(i|\theta) & \text{if } \theta_i \in M(m), \\ 0 & \text{if } \theta_i \notin M(m). \end{cases} \tag{4}$$

This will provide a different perspective regarding how an **M**-mechanism works.

3.2 Example

Before proceeding further, we illustrate the two main objectives of the analysis. For the example, consider two buyers $I \equiv \{1, 2\}$ and a symmetric and uniform distribution, $F_A = F_B = U[0, 1]$. With the symmetry, we just examine good A. Suppose that the seller adopts a member criterion $m(\theta_i) = \max\{v_i, w_i\} - \frac{1}{2}$ as in Figure 1, and that buyer 1 is a member such that his reported type satisfies the criterion $\theta_1 \in M(m)$ or $\max\{v_1, w_1\} \geq \frac{1}{2}$. We compare two different mechanisms, and the first mechanism is called the canonical second-price auction (SPA) **M**-mechanism, since membership naturally applies to a standard SPA as follows.[19]

[19]It is easy to construct a mechanism to illustrate how a member set can interfere with the Bayesian incentive compatibility – as well as the dominant incentive compatibility. First, with a positive membership fee $e > 0$, an **M**-mechanism is not dominant incentive compatible. Second, let us mix match the alternative

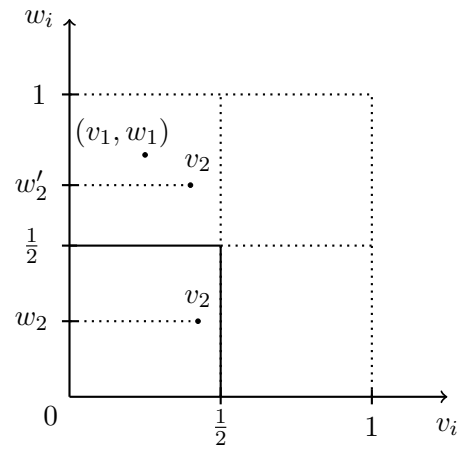

Figure 1: Membership of Example 1

Example 1 (two bidders and symmetric & uniform distribution) The canonical SPA **M**-mechanism's allocation rule q_A^M is given by

$$q_A^M(1|\theta_1, \theta_2) = \begin{cases} 1 & \text{if } v_1 > v_2 \text{ or } \theta_2 \notin M(m), \\ 0 & \text{if } v_1 < v_2, \theta_2 \in M(m), \end{cases}$$

and its transfer rule t_A^M is given by

$$t_A^M(1|\theta_1, \theta_2) = \begin{cases} v_2 & \text{if } v_1 > v_2,\ \theta_2 \in M(m), \\ 0 & \text{if } v_1 < v_2 \text{ or } \theta_2 \notin M(m), \end{cases}$$

with the standard tie-breaking rule (that is, winning and transfer with equal probability if $v_1 = v_2$). For the canonical SPA **M**-mechanism, if buyer 2 is a member, buyer 1's winning only depends on valuations of good A, not those of the other good, and if buyer 2 is not a member, buyer 1 wins good A with zero payment. The two aspects appear natural, but they are in fact *restraining* the seller's power such that $p_A(i|\theta)$ and $x_A(i|\theta)$ in (3) are restricted to $p_A(i|v)$ and $x_A(i|v)$ and such that $p_B(i|\theta)$ and $x_B(i|\theta)$ in (4) are restricted to $p_B(i|w)$ and $x_B(i|w)$. Then, given the canonical mechanism, by reporting v_1', buyer 1 with v_1 obtains the interim payoff from good A such that

$$[\Pr(\theta_2 \notin M(m)) + \Pr(\theta_2 \in M(m), V_2 < v_1')]\, v_1 - \Pr(\theta_2 \in M(m), V_2 < v_1')\mathbb{E}[v_2|\theta_2 \in M(m), V_2 < v_1'].$$

SPA **M**-mechanism's allocation rule $\widetilde{q}_A^M$ and the canonical SPA **M**-mechanism's payment rule t_A^M in Example 1. Then, buyer 1's interim payoff from reporting v_1' for good A is $v_1'v_1 - \Pr(\theta_2 \in M(m), V_2 < v_1')\mathbb{E}[v_2|\theta_2 \in M(m), V_2 < v_1'] = \begin{cases} v_1'v_1 - \frac{1}{4}{v_1'}^2 & \text{if } v_1' < \frac{1}{2}, w_1' > \frac{1}{2}, \\ v_1'v_1 - \frac{1}{2}{v_1'}^2 + \frac{1}{16} & \text{if } v_1' > \frac{1}{2}. \end{cases}$ Thus, if his type v_1 is less than $\frac{1}{2}$, buyer 1 has an *over-reporting* incentive. If his type $v_1 < \frac{1}{2}$, from the intersection of the two curves from the interim payoff, the kinked point $\frac{1}{2}$ is a unique global max, whereas if $v_1 > \frac{1}{2}$, the whole curve is strictly increasing, even including the kinked point, on $v_1' < v_1$, making the truthful report $v_1' = v_1$ a unique global max.

This mechanism satisfies incentive compatibility, and buyer 1's payment is given by[20]

$$\Pr(\theta_2 \in M(m), V_2 < v_1)\mathbb{E}[v_2|\theta_2 \in M(m), V_2 < v_1] = \begin{cases} \frac{1}{4}{v_1}^2 & \text{if } v_1 < \frac{1}{2}, w_1 > \frac{1}{2}, \\ \frac{1}{2}{v_1}^2 - \frac{1}{16} & \text{if } v_1 > \frac{1}{2}. \end{cases}$$

Then, the revenue from a single buyer is $\int_{1/2}^{1} \left(\frac{1}{2}{v_1}^2 - \frac{1}{16}\right) dv_1 + \frac{1}{2}\int_0^{1/2} \left(\frac{1}{4}{v_1}^2\right) dv_1 = \frac{23}{192}$.

Despite the natural appeal of the mechanism, let us examine whether we can improve it. Consider an alternative **M**-mechanism that consists of an allocation rule $\widetilde{q}_A^M$ given by

$$\widetilde{q}_A^M(1|\theta_1, \theta_2) = \begin{cases} 1 & \text{if } v_1 > v_2, \\ 0 & \text{if } v_1 < v_2, \end{cases}$$

and a transfer rule $\widetilde{t}_A^M$ given by

$$\widetilde{t}_A^M(1|\theta_1, \theta_2) = \begin{cases} v_2 & \text{if } v_1 > v_2, v_2 \geq \frac{1}{2}, \\ 2v_2 & \text{if } v_1 > v_2, v_2 < \frac{1}{2}, w_2 > \frac{1}{2}, \\ 0 & \text{if } v_1 < v_2 \text{ or } \theta_2 \notin M(m). \end{cases}$$

The key difference between the canonical SPA **M**-mechanism and the alternative SPA **M**-mechanism is that for the latter, even if buyer 2 is not a member, buyer 1's winning depends on v_2, as illustrated in Figure 1 with (v_2, w_2) and (v_2, w_2'), and his transfer changes depending on where buyer 2's type is located.[21] In other words, in contrast to the canonical mechanism in which once buyer i enters, good k's allocation and payment depend on only valuations of the relevant good, the alternative mechanism empowers the seller so that one good's allocation and payment depend on the other good's valuations, and additionally, a participant's allocation and payment depend on a non-participant's valuations. Then, regardless of where buyer 1's reported type is, his interim payment is always $\frac{1}{2}v_1^2$, so buyer 1's interim payoff from reporting v_1' for good A is $v_1'v_1 - \frac{1}{2}{v_1'}^2$, satisfying incentive compatibility, which yields the revenue from a single buyer $\int_0^1 \left(\frac{1}{2}v_1^2\right) dv_1 = \frac{1}{6} = \frac{32}{192}$ greater than the revenue above.

This exercise shows that the alternative SPA **M**-mechanism dominates the canonical SPA **M**-mechanism (both have the same member set and, therefore, collect the same membership fee). For the dominant allocation rule, despite a member set, we can allocate a good as though we "ignore" it. At this point, we do not know whether allocating a good in that way is optimal. This is our first objective, the characterization of the optimal mechanism *among*

[20]The former expected payment is derived from double integrals $\int_0^{v_1}\int_{\frac{1}{2}}^1 v_2 dw_2 dv_2$, and the latter from $\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1 v_2 dw_2 dv_2 + \int_{\frac{1}{2}}^{v_1}\int_0^1 v_2 dw_2 dv_2$. Regarding the notation, having prime like v_1' is not a generic notation for reporting, following the convention in the literature; that is, it depends on each context.

[21]That is, for a mechanism, even a non-member *reports*, so allocating good A following the alternative SPA **M**-mechanism, $q_A(1|\theta_1, \theta_2) = 0$ from $v_2 > v_1$, involves no problem, even when buyer 1 is only a bidder. However, for its implementation as an auction setting, a non-member simply does not make a bid, so to implement the same winning probability as the mechanism, the indirect *auction* counterpart of the mechanism needs to be reconstructed with a random reserve price as in Section 7.

M-mechanisms (Theorem 1), and the next question is then to compare its performance with the optimal no-membership mechanism by Myerson (1981) (Theorems 2 and 3). For the second main goal, a membership fee plays a key role, as will be shown.

3.3 Incentive compatibility and individual rationality

Now, we provide incentive compatibility and individual rationality conditions with membership. Let θ_{-i} be a vector of all buyers' types except for buyer i's, as an element of $\Theta_{-i} \equiv \times_{j\in I\setminus\{i\}}\Theta_j$, and, similarly, let v_{-i} and w_{-i} be defined, and denote their CDFs by $\mathcal{F}_A(v_{-i}) \equiv \times_{j\in I\setminus\{i\}}F_A(v_j)$ and $\mathcal{F}_B(w_{-i}) \equiv \times_{j\in I\setminus\{i\}}F_B(w_j)$. Then, the expected probabilities that buyer i obtains good A and B, respectively, are defined as $Q_A^M(\theta_i) \equiv \int_{\Theta_{-i}} q_A^M(i|\theta)d\mathcal{F}_A(v_{-i})\times\mathcal{F}_B(w_{-i})$ and $Q_B^M(\theta_i) \equiv \int_{\Theta_{-i}} q_B^M(i|\theta)d\mathcal{F}_A(v_{-i})\times\mathcal{F}_B(w_{-i})$. The expected payment that member i makes to the seller for good k is $T_k^M(\theta_i) \equiv \int_{\Theta_{-i}} t_k^M(i|\theta)d\mathcal{F}_A(v_{-i}) \times \mathcal{F}_B(w_{-i})$, and lastly denote $T^M(\theta_i) \equiv T_A^M(\theta_i) + T_B^M(\theta_i)$. Buyer i's interim expected payoff if he is a member and reports his type truthfully is

$$u(\theta_i) \equiv v_i Q_A^M(\theta_i) + w_i Q_B^M(\theta_i) - T^M(\theta_i) - e, \tag{5}$$

and denote by $u_0 \geq 0$ the payoff that buyer i obtains if he is not a member. We denote a non-member's truthful payoff as a constant, without loss of generality, since given $q_A(i|\theta) = q_B(i|\theta) = 0$, two different non-member types must have the same payoff to satisfy the incentive compatibility – a non-member's incentive not to misreport the other non-member-type. It is clear that any revenue-maximizing mechanism chooses $u_0 = 0$.[22]

An **M**-mechanism is said to be *incentive compatible* (IC) if for each $\theta_i, \theta_i' \in M(m)$,

$$\begin{aligned} v_i Q_A^M(\theta_i) - T_A^M(\theta_i) &\geq v_i Q_A^M(\theta_i') - T_A^M(\theta_i'), \\ w_i Q_B^M(\theta_i) - T_B^M(\theta_i) &\geq w_i Q_B^M(\theta_i') - T_B^M(\theta_i'), \end{aligned} \tag{6}$$

and for each $\theta_i \in \Theta_i \setminus M(m)$,

$$u_0 = 0 \geq v_i Q_A^M(\theta_i') + w_i Q_B^M(\theta_i') - T^M(\theta_i') - e \text{ for all } \theta_i' \in M(m). \tag{7}$$

The condition (6), the first set of IC, is called the *member incentive compatibility* (MIC) – no incentive for a member to misreport the other member-type. Note that MIC is given as the separable incentive compatibility conditions. To show this, we can simply apply the same gist of the revelation principle: Given a collection of Bayesian Nash equilibria $(\boldsymbol{\beta}_A, \boldsymbol{\beta}_B)$ from an indirect mechanism, there exists a direct truthful mechanism that implements the same equilibrium strategy profile and payoffs as those from the indirect mechanism. It is straightforward to find that this *extended revelation principle* for a membership mechanism with two Bayesian Nash equilibria for members requires two *separable* incentive compatibility

[22] Specifically, given zero allocation in (2), for each $\theta_i \in \Theta_i \setminus M(m)$, $t^M(\theta_i)$ is chosen to satisfy $T^M(\theta_i) = 0$ for each revenue-maximization among $T^M(\theta_i) \geq 0$ satisfying a non-member's individual rationality $u_0 \geq 0$.

conditions for the two goods.[23] That is, simply, by taking a contrapositive of it, if *one of* the IC in (6) is violated, then a Bayesian Nash equilibrium for that good in an indirect mechanism is violated, having a contradiction with the revelation principle. Despite the separable IC, the lowest type to participate depends on a member set and so a decision to become a member – in addition to a multidimensional type θ_i in (6). An incentive for a member not to "pretend" to be a non-member is satisfied by the individual rationality condition below. On the other hand, the condition (7), the second part of IC, is a *non*-member's incentive compatibility: no incentive to misreport to become a member.

An **M**-mechanism is said to be *individually rational* (IR) if for each $\theta_i \in M(m)$,

$$u(\theta_i) \geq 0. \tag{8}$$

A non-member's individual rationality condition can be ignored from $u_0 = 0$.

One may argue that an **M**-mechanism's mapping $m(\cdot)$ and its membership fee e can be chosen in such a way that a member set is identical to the entire type set, $M(m) = \Theta_i$ (*e.g.*, $m(\theta_i) = a$ for any non-negative constant a, and $e = 0$). Then, given the buyer's separate purchase, a membership becomes separate participation by simply "duplicating" the result by Myerson (1981); thus, the definition of an **M**-mechanism includes the standard approach. We provide the main results for *pure membership*, *i.e.*, membership with *no ex-ante reserve price or absolute exclusion* for $M(m)$ because the dominance of pure membership is stronger, not to mention the simplicity.[24] Yet, this simple approach does not limit the generality of the dominance since the main results can be transformed neatly into an **M**-mechanism with ex-ante reserve prices, the full-fledged model in Section 7.[25] Then, by adding ex-ante reserve prices to an **M**-mechanism, the membership's dominance over separate participation can be simply restated: The "effective" **M**-membership with reserve prices such that $\Theta_i \setminus M(m) \neq \emptyset$

[23] An allocation rule for good A is denoted by $y_A^M : \mathbb{R}_+^N \to \Delta(X_A)$, and an allocation rule for good B is $y_B^M : \mathbb{R}_+^N \to \Delta(X_B)$. Additionally, a payment rule for good A is denoted by $z_A^M : \mathbb{R}_+^N \to \mathbb{R}^N$, and a payment rule for good B is $z_B^M : \mathbb{R}_+^N \to \mathbb{R}^N$, where $y_k^M(i|\cdot)$ and $z_k^M(i|\cdot)$ denote an allocation and a payment rule of good k for buyer i. Then, for each i, by defining composite mappings such as $q_A^M(i|\theta) \equiv y_A^M(i|\boldsymbol{\beta}_A(\theta))$, $t_A^M(i|\theta) \equiv z_A^M(i|\boldsymbol{\beta}_A(\theta))$, $q_B^M(i|\theta) \equiv y_B^M(i|\boldsymbol{\beta}_B(\theta))$, and $t_B^M(i|\theta) \equiv z_B^M(i|\boldsymbol{\beta}_B(\theta))$, the extended revelation principle leads to the separable incentive compatibility conditions.

[24] Although a reserve price may not be a proper term for a direct mechanism, we use it due to its widely recognized usage. For the readers seeking for a precise definition, from the perspective of the multi-dimensional aspect, such no assignment rule can be defined formally as *absolute exclusion* such that a mechanism $(q, t_1, ..., t_N)$ is said to satisfy absolute exclusion for $\widehat{\Theta}_i \subseteq \Theta_i$ if there exists $i \in I$ such that for each $\theta \in \Theta$, $q_A(i|\theta) = 0$ or $q_B(i|\theta) = 0$ for some $\theta_i \in \widehat{\Theta}_i \cap \mathring{\Theta}_i$, where $\mathring{\Theta}_i$ denotes the interior of the type set Θ_i. That is, the exclusion applies to buyer i only given his reported type, *regardless of* any reports by other buyers. Thus, for instance, in an auction, the absolute exclusion is simply a direct mechanism of two reserve prices for two items.

[25] An additional and practical point to consider, as will be discussed in Section 7, is related to secret reserve prices, a well-established tool in reality that can implement the optimal allocation with membership in Section 4. Thus, if we adopt ex-ante reserve prices, in addition to secret reserve prices, the mechanism with both types of reserve price might not be a convincing real indirect institution.

dominates the latter, non-effective **M**-mechanism; put differently, a membership with a positive fee $e > 0$ dominates the duplication of Myerson's solution.

3.4 Revenue

The seller chooses one among mechanisms satisfying IC with (6) & (7) and IR (8). We first simplify the seller's choice set. This procedure enables us to replace a non-member's incentive compatibility (7) and (IR) with a single condition. We call it *boundary condition* (BC) since the condition holds if IR in (8) is binding such that all types in the minimal member criterion $\overline{M}(m)$ must have zero payoff.[26] Formally, BC is said to be satisfied if

$$u(\theta_i) = 0 \text{ for each } \theta_i \in \overline{M}(m). \tag{9}$$

Overall, by the definition of BC and the lemma below, we replace the incentive compatibility and individual rationality of an **M**-mechanism with MIC in (6) and BC in (9).

Lemma 1 *Any revenue-maximizing **M**-mechanism $\Gamma = (q^M, t^M, m, e)$ is incentive compatible and individually rational if and only if it satisfies MIC and BC.*

Lemma 1, on the other hand, poses a challenge to the *existence* of a member set, given the following consequence with equality:

$$M(m) = \{\theta_i \in \Theta_i : u(\theta_i) \geq 0\}. \tag{10}$$

A member set $M(m)$ may constrain an allocation rule q^M and a transfer rule t^M – to meet the condition in (2) together with MIC and BC – and a corresponding interim payoff $u(\theta_i)$ as well. This in turn results in a set of participating types $\{\theta_i \in \Theta_i : u(\theta_i) \geq 0\}$ in the right-hand side above, which must be identical to the initially chosen member set in the left-hand side. Hence, the analysis, in nature, involves a *fixed point* type argument: A member set generates a mechanism, and the mechanism must "duplicate" the member set. These potential complications, however, eventually disappear thanks to the characterization of the optimal allocation in the first main result of the next section.

The seller's revenue from a single buyer with a member set is given as

$$R_M(\Gamma) \equiv \int_{M(m)} [T_A(\theta_i) + T_B(\theta_i) + e]\, dF_A(v_i) \times F_B(w_i), \tag{11}$$

and an **M**-mechanism is optimal if it maximizes the above revenue subject to MIC in (6) and BC in (9). To analyze the mechanism for each dimension, it is convenient to have two different configurations for the *minimal member qualification* such as

$$v_e(w_i) \equiv \min\{v_i \in [\underline{v}, \overline{v}] : \theta_i \in M(m)\}, w_e(v_i) \equiv \min\{w_i \in [\underline{w}, \overline{w}] : \theta_i \in M(m)\}. \tag{12}$$

[26] A non-member obtains zero as well like a member type on the member standard line, but recall that $u(\theta_i)$ denotes a *member*'s interim payoff, especially including a membership fee e.

To be clear, a member standard line $\overline{M}(m)$ is a *subset* of the union of $v_e(w_i)$ and $w_e(v_i)$. That is, given a fixed dimension, either $v_e(w_i)$ or $w_e(v_i)$ is the other dimension that is required to be minimally qualified as a member, which can be "active," greater than the lowest valuation for either item, $\underline{v}$ or $\underline{w}$, or not, depending on a chosen w_i or v_i. For instance, from Example 1, $v_e(w_i)$ is given as

$$v_e(w_i) = \begin{cases} \frac{1}{2} & \text{if } w_i < \frac{1}{2}, \\ 0 & \text{if } w_i \geq \frac{1}{2}, \end{cases} \tag{13}$$

where if $w_i > \frac{1}{2}$, $v_e(w_i)$ is *not on* the member standard line $\overline{M}(m)$, but equal to $\underline{v} = 0$. We can also find suprema $\widehat{v}_e \equiv \sup\{v_i \in [\underline{v}, \overline{v}] : w_e(v_i) > \underline{w}\}$ and $\widehat{w}_e \equiv \sup\{w_i \in [\underline{w}, \overline{w}] : v_e(w_i) > \underline{v}\}$, satisfying $w_e(\widehat{v}_e) = \underline{w}$ and $v_e(\widehat{w}_e) = \underline{v}$. With the suprema identifying each dimension's intercept for the member standard line, (9) from BC can be restated: for each $v_i \in [\underline{v}, \widehat{v}_e], u(v_i, w_e(v_i)) = 0$ and for each $w_i \in [\underline{w}, \widehat{w}_e], u(v_e(w_i), w_i) = 0$.

We first find the optimal membership mechanism before comparing bundling participation and separate participation.

4 Optimal membership mechanism

In this section, we characterize the optimal mechanism given membership. In Myerson (1981), for a seller's revenue maximization, the lowest valuation obtains the lowest payoff. In particular, it is convenient to pin down the lowest valuation's payoff *before* a mechanism characterizes an optimal allocation. As well known, this procedure transforms a set of mechanisms satisfying incentive compatibility and individual rationality into a seller's simple and tractable choice set with increasing expected winning probability. However, in this multi-dimension case, with an **M**-mechanism, a type with the lowest valuation in one dimension can have a high valuation in the other dimension; therefore, it is not immediately apparent whether the lowest valuation in one dimension attains the lowest payoff or not. For instance, in (13), if $w_i > \frac{1}{2}$, the lowest v_i to be a member $v_e(w_i)$ is *not* on the member standard line $\overline{M}(m)$, which means that $(v_e(w_i), w_i)$ can have some positive payoffs.

Hence, we proceed in this section with the following strategy. First, we show the separability & independence of allocation rules for both goods. Then, unlike the standard procedure, we consider a seller's choice set *larger* than **M**-mechanisms satisfying incentive compatibility and individual rationality via MIC and BC, because in this step, for **M**-mechanisms, we cannot show the equivalence relationship between a simple choice set and such mechanisms. For the larger set, however, by showing the differentiability of the optimal allocation rules, we obtain the equivalence in the *aftermath*.

4.1 Separability & independence of allocation

We say that an **M**-mechanism allocates two goods separably and independently on $M(m)$ if an *interim allocation* for a good depends only on valuations of *that* good on $M(m)$.[27] The separability and independence of an allocation rule of a membership mechanism holds.[28]

Proposition 1 *An incentive compatible **M**-mechanism $\Gamma = (q^M, t^M, m, e)$ allocates two goods separably and independently on $M(m)$ such that*

(i) for each pair $(v_i, w_i) \neq (v_i, w_i') \in M(m), Q_A^M(v_i, w_i) = Q_A^M(v_i, w_i')$ for almost all v_i.

(ii) for each pair $(v_i, w_i) \neq (v_i', w_i) \in M(m), Q_B^M(v_i, w_i) = Q_B^M(v_i', w_i)$ for almost all w_i.

The separability and independence of an allocation rule on $M(m)$ are by and large straightforwardly derived, based on two separable incentive compatibility conditions in (6). By Proposition 1, with slight abuse of notations, denote $Q_A^M(v_i) = Q_A^M(v_i, w_i) = Q_A^M(v_i, w_i')$ for each $w_i \neq w_i'$ and $Q_B^M(w_i) = Q_B^M(v_i, w_i) = Q_B^M(v_i', w_i)$ for each $v_i \neq v_i'$.

Importantly, with this result, we define the following probability given $w_e(v_i)$:

$$\widetilde{q}_A^M(i|v) \equiv \int_{[\underline{w},\overline{w}]^{N-1}} q_A^M(i|v_i, w_e(v_i), v_{-i}, w_{-i}) d\mathcal{F}_B(w_{-i}), \tag{14}$$

with a profile of valuations for good A, $v \equiv (v_1, ..., v_N)$ – the intention is not to underestimate the seller's power to control w_{-i} in an **M**-mechanism, as will be shown. Similarly, given $v_e(w_i)$, we can define $\widetilde{q}_B^M(i|w) \equiv \int_{[\underline{v},\overline{v}]^{N-1}} q_B^M(i|v_e(w_i), w_i, v_{-i}, w_{-i}) d\mathcal{F}_A(v_{-i})$, with a profile for good B, $w \equiv (w_1, ..., w_N)$. The modified allocation rule $\widetilde{q}_A^M(i|v)$ for good A (resp. $\widetilde{q}_B^M(i|w)$ for good B) is the expected probability that the seller assigns good A (resp. good B) to buyer i *only* conditional on reports v (resp. w) for that good. With them, $Q_A^M(v_i)$ and $Q_B^M(w_i)$ can be rewritten as $Q_A^M(v_i) = \int_{[\underline{v},\overline{v}]^{N-1}} \widetilde{q}_A^M(i|v) d\mathcal{F}_A(v_{-i})$ and $Q_B^M(w_i) = \int_{[\underline{w},\overline{w}]^{N-1}} \widetilde{q}_B^M(i|w) d\mathcal{F}_B(w_{-i})$.

4.2 Optimal allocation rule for a larger choice set

We start by defining type θ_i member's payoff from good A and B, respectively, as

$$u_A(v_i, w_i) \equiv v_i Q_A^M(v_i) - T_A^M(v_i, w_i) \text{ and } u_B(v_i, w_i) \equiv w_i Q_B^M(w_i) - T_B^M(v_i, w_i). \tag{15}$$

By Lemma 1, a set of **M**-mechanisms satisfying IC and IR is equivalent to a set of **M**-mechanisms satisfying MIC and BC. Given MIC together with the minimal qualifications

[27] Unlike the independence, the separability of either rule is immediate. Any joint probability for allocation of two items can yield two separate marginal probabilities for them, as demonstrated in the model section, so the separability of an allocation rule follows from the property of a joint probability. The separability of a transfer rule follows from the separate purchase.

[28] In one good's allocation, the other good's valuation does not affect a buyer's payoff directly, so in that sense, for the proof, the other good's valuation can essentially be treated similarly as a non-contractible quality as Yoo (2016).

$v_e(w_i)$ and $w_e(v_i)$ in (12), the standard procedure yields that if a mechanism satisfies MIC, (i) $Q_A^M(v_i)$ and $Q_B^M(w_i)$ are increasing, and (ii) $u_A(v_i, w_i) = u_A(v_e(w_i), w_i) + \int_{v_e(w_i)}^{v_i} Q_A^M(x)dx$ and $u_B(v_i, w_i) = u_B(v_i, w_e(v_i)) + \int_{w_e(v_i)}^{w_i} Q_B^M(y)dy$, where recall that $u(v_i, w_e(v_i)) = 0$ for *only* $v_i \in [\underline{v}, \widehat{v}_e]$ and $u(v_e(w_i), w_i) = 0$ for *only* $w_i \in [\underline{w}, \widehat{w}_e]$, respectively – otherwise, they can have positive values. Since the *monotonicity* of the mapping $m(\cdot)$ implies that $\int_{\underline{v}}^{v_e(w_i)} Q_A^M(x)dx = 0$ and $\int_{\underline{w}}^{w_e(v_i)} Q_B^M(y)dy = 0$, the equivalence still holds even when the lower bound, *i.e.*, the minimal qualification, in each good is *stretched* to the corresponding lowest feasible valuation, $\underline{v}$ or $\underline{w}$. This is summarized as follows.

If an **M**-mechanism satisfies MIC, for each $i \in I$ and every $\theta_i \in \Theta_i$,

(i) $Q_A^M(v_i)$ and $Q_B^M(w_i)$ are increasing;

(ii) $u_A(v_i, w_i) = u_A(v_e(w_i), w_i) + \int_{\underline{v}}^{v_i} Q_A^M(x)dx, u_B(v_i, w_i) = u_B(v_i, w_e(v_i)) + \int_{\underline{w}}^{w_i} Q_B^M(y)dy.$ (16)

The relationship holds only for necessity, as written above at this step; thus, we cannot characterize the seller's choice set for mechanisms satisfying IC through MIC in a standard way. To see why the sufficiency may not hold, we consider good A and derive the interim payment for good A from (ii) as

$$T_A^M(v_i, w_i) = v_i Q_A^M(v_i) - \int_{\underline{v}}^{v_i} Q_A^M(x)dx - u_A(v_e(w_i), w_i). \tag{17}$$

Then, if a buyer reports his type truthfully, the payoff is $\int_{\underline{v}}^{v_i} Q_A^M(x)dx + u_A(v_e(w_i), w_i)$, whereas if he misreports, it is $v_i Q_A^M(v_i') - \left[v_i' Q_A^M(v_i') - \int_{\underline{v}}^{v_i'} Q_A^M(x)dx - u_A(v_e(w_i'), w_i')\right]$. The difference yields

$$\int_{v_i}^{v_i'} Q_A^M(v_i')dx - \int_{v_i}^{v_i'} Q_A^M(x)dx + u_A(v_e(w_i), w_i) - u_A(v_e(w_i'), w_i'). \tag{18}$$

As well known from the single dimension problem, without the term $u_A(v_e(w_i'), w_i') - u_A(v_e(w_i), w_i)$, the monotonicity of Q_A^M yields the incentive compatibility, satisfying the sufficiency.[29] However, in this multi-dimension case, with the term, the monotonicity of Q_A^M is not sufficient for the incentive compatibility; in particular, the monotonicity of Q_A^M can fail in "compensating" for the difference, $u_A(v_e(w_i'), w_i') - u_A(v_e(w_i'), w_i')$ so that the difference $u_A(v_e(w_i), w_i) \neq u_A(v_e(w_i), w_i)$ *interferes* with the incentive compatibility, or it can succeed in compensating for that. Thus, to probe further whether any small difference $u_A(v_e(w_i), w_i) \neq u_A(v_e(w_i'), w_i')$ can have such a conflict with the incentive compatibility, we must rely on an infinitesimal change of Q_A^M, *i.e.*, its *differentiability*. With the differentiability, we show that the incentive compatibility, in fact, requires $u_A(v_e(w_i), w_i) = u_A(v_e(w_i'), w_i')$ for $w_i \neq w_i'$, and thereby prove

[29] That is, $\int_{v_i}^{v_i'} Q_A^M(v_i')dx - \int_{v_i}^{v_i'} Q_A^M(x)dx \geq 0$; see, for example, Börgers (2015).

the sufficiency. This means that it is necessary to characterize an optimal allocation rule *before* we pin down the term $u_A(v_e(w_i), w_i)$, unlike in a single dimension.

From the interim payment of (17), the expected revenue from buyer i regarding good A's sale is given as

$$\begin{aligned}\Pr\big(\theta_i \in M(m)\big)\mathbb{E}\left[T_A^M(v_i)|\theta_i \in M(m)\right] &= \int_{M(m)} T_A^M(v_i, w_i) dF_A(v_i) \times F_B(w_i) \qquad (19)\\ &= \int_{M(m)} \Big[v_i Q_A^M(v_i) - \int_{\underline{v}}^{v_i} Q_A^M(x)dx\Big] dF_A(v_i) \times F_B(w_i)\\ &\quad - \int_{M(m)} u_A(v_e(w_i), w_i) dF_A(v_i) \times F_B(w_i).\end{aligned}$$

An identical procedure applies to the interim payment for good B, T_B^M. Since the second term does not depend on an allocation rule for good A, we first focus on the first term. By Fubini's theorem and the independence of two random variables from F_A and F_B, we find the probability that v_i is smaller than x, given the member set $M(m)$, which is denoted by $H_A^M(x) \equiv \Pr(v_i \leq x, \theta_i \in M(m))$, such that

$$\begin{aligned}H_A^M(x) &= \int_{M(m):v_i \leq x} dF_A(v_i) \times F_B(w_i) \qquad (20)\\ &= \begin{cases} \int_{\underline{v}}^{x} [1 - F_B(w_e(v_i))] dF_A(v_i) & \text{if } x < \widehat{v}_e,\\ \int_{\underline{v}}^{\widehat{v}_e} [1 - F_B(w_e(v_i))] dF_A(v_i) + F_A(x) - F_A(\widehat{v}_e) & \text{if } x \geq \widehat{v}_e.\end{cases}\end{aligned}$$

Note that with H_A^M, a conditional distribution $\Pr(v_i \leq x \mid \theta_i \in M(m))$ can be constructed as $\Pr(v_i \leq x \mid \theta_i \in M(m)) = \frac{H_A^M(x)}{\Pr(\theta_i \in M(m))}$ and that each of $H_A^M(x)$ and $H_B^M(y)$ is differentiable in the two intervals above, respectively.[30] We emphasize that H_A^M is defined for the event that bidder i is participating – so the seller can collect the payment from bidder i – *not* for bidder i's *beliefs* about the event that the other bidders participate; the latter must be incorporated into $Q_A^M(\cdot)$, which can be intractably complicated by considering combinations for counting how many buyers participate and with what probabilities.

Denote by h_A^M the derivative of $H_A^M(x)$. Then, by the standard procedure, the first term of each buyer's revenue from good A in (19) can be rewritten as

$$\begin{aligned}\Pr\big(\theta_i \in M(m)\big)\mathbb{E}\Big[v_i Q_A^M(v_i) - \int_{\underline{v}}^{v_i} Q_A^M(x)dx \mid v_i \in M(m)\Big] &= \int_{\underline{v}}^{\overline{v}} \Big[v_i Q_A^M(v_i) - \int_{\underline{v}}^{v_i} Q_A^M(x)dx\Big] dH_A^M(v_i)\\ &= \int_{\underline{v}}^{\overline{v}} Q_A^M(v_i)\psi_A^M(v_i) h_A^M(v_i) dv_i,\end{aligned}$$

where we define the **M**-mechanism's *virtual valuation* function as

$$\psi_A^M(x) \equiv x - \frac{H_A^M(\overline{v}) - H_A^M(x)}{h_A^M(x)}. \qquad (21)$$

[30] That is, $w_e(v_i)$ and $v_e(w_i)$ might not be continuous, as in (13) from Example 1, but if either $w_e(v_i)$ or $v_e(w_i)$ is discontinuous, it is discontinuous only at the break point, either $\widehat{v}_e$ or $\widehat{w}_e$; that is, it is continuous, and furthermore differentiable in the interval, $x < \widehat{v}_e$, and in the other interval, $x \geq \widehat{v}_e$, separably.

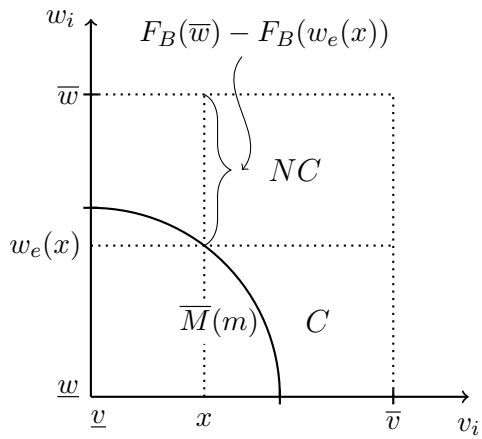

Figure 2: Sufficiency for the modified monotonicity if F_k is not convex

A distribution H_A^M for good A's valuations is said to be "regular" if the modified virtual valuation function in (21) is strictly increasing (similarly for good B). Now, the revenue in (11) can be reformulated as

$$R_M(\Gamma) = \int_{\underline{v}}^{\overline{v}} Q_A^M(v_i)\psi_A^M(v_i)dH_A^M(v_i) + \int_{\underline{w}}^{\overline{w}} Q_B^M(w_i)\psi_B^M(w_i)dH_B^M(w_i) + eH_A^M(\overline{v}), \qquad (22)$$

where the last term is the expected membership fee payment, and the probability of participating in membership is the same regardless of what to integrate, *i.e.*, $H_A^M(\overline{v}) = H_B^M(\overline{w})$. Note again that we are solving the optimal mechanism subject to a larger choice set in the sense that it is given as a set of mechanisms satisfying increasing Q_k^M for all k (i) and the payment rule T_k^M determined following (ii) of (16) together with BC, which *includes* a set of mechanisms satisfying MIC and BC.

With the regularity, the first main result, following the lemma below, shows that the optimal mechanism assigns good A (good B) to the buyer with the highest valuation for good A (good B). For each $k \in \{A, B\}$, the regularity of a modified cumulative distribution H_k^M is satisfied for a large class of distributions of F_k.

Lemma 2 *For each $k \in \{A, B\}$, if F_k is convex, H_k^M is regular for each $M(m)$. If F_A is not convex, then H_A^M is regular under the condition $\int_x^{\overline{v}} \left[1-2F_B(w_e(x))+F_B(w_e(v_i))\right] f_A(v_i)dv_i > 0$ for all x. Similarly, if F_B is not convex, then H_B^M is regular under $\int_y^{\overline{w}} \left[1 - 2F_A(v_e(y)) + F_A(v_e(w_i))\right] f_B(w_i)dw_i > 0$ for all y.*

In the standard single item case with a symmetric valuation distribution F, the monotone hazard rate ensures that the virtual valuation is (strictly) increasing so that the optimal mechanism allocates the item to the highest valuation with probability 1, and a sufficient condition for the property to hold is the convexity of F or $f' \geq 0$; that is, if the distribution is increasing at an increasing rate, the virtual valuation is increasing. Even with a member set $M(m)$ given two dimensions, such convexity of good k's valuation distribution results in

the increasing modified virtual valuation. If the convexity is not satisfied in the lemma, on the other hand, the sufficient condition utilizes the monotone hazard rate condition $f_k'(x) \geq -\frac{f_k(x)^2}{1-F_k(x)}$. To understand its role, we rewrite the condition for good A (similarly for good B) as

$$\int_x^{\overline{v}} \left[F_B(\overline{w}) - F_B(w_e(x))\right] f_A(v_i)dv_i > \int_x^{\overline{v}} \left[F_B(w_e(x)) - F_B(w_e(v_i))\right] f_A(v_i)dv_i \text{ for all } x,$$

which requires that the integral of the non-constrained area (NC) dominates that of the constrained area (C) by membership, as depicted in Figure 2, so that changes in $H_A^M(\overline{v}) - H_A^M(x)$ *emulate* changes in $1 - F_A(x)$, satisfying the monotone hazard rate.

The first main result shows that given the regularity, the membership does not alter the nature of the standard revenue-maximizing mechanism: the optimal mechanism allocates good $k \in \{A, B\}$ to the member with the highest valuation for that good, resulting in winning probabilities for good A and B, $G_A(v_i) \equiv F_A(v_i)^{N-1}$ and $G_B(w_i) \equiv F_B(w_i)^{N-1}$, respectively.[31]

Theorem 1 *Suppose H_k^M is regular for all $k \in \{A, B\}$. Among all **M**-mechanisms satisfying (16) and BC, those **M**-mechanisms maximizing the revenue choose the allocation such that*

$$\widetilde{q}_A^M(i|v) = \begin{cases} 1 & \text{if } v_i > v_j \text{ for all } j \in I \text{ and } j \neq i, \\ 0 & \text{otherwise;} \end{cases}$$

$$\widetilde{q}_B^M(i|w) = \begin{cases} 1 & \text{if } w_i > w_j \text{ for all } j \in I \text{ and } j \neq i, \\ 0 & \text{otherwise,} \end{cases}$$

Moreover, the allocation rule implies that for any two member sets $M \neq M'$, $Q_A^M(v_i) = Q_A^{M'}(v_i) = G_A(v_i)$ and $Q_B^M(w_i) = Q_B^{M'}(w_i) = G_B(w_i)$.

The regularity condition in Lemma 2 still results in the simple optimal allocation rule, regardless of a member set. It is somewhat surprising to find that a simple condition, convexity of distribution, engenders such an outcome. The potentially complex existence problem of a member set, based on a fixed-point type argument, is resolved from the increasing modified virtual valuation: for each $M \neq M'$, $Q_A^M(v_i) = Q_A^{M'}(v_i) = G_A(v_i)$ and $Q_B^M(w_i) = Q_B^{M'}(w_i) = G_B(w_i)$, *regardless of* where a type θ_i is (recall, for instance, the alternative SPA **M**-mechanism in Example 1). The optimal allocation resembling the single dimension's optimal allocation, however, in fact has deeper implications on the actual allocation – though it appears simple. A two-buyer case is sufficient to illuminate the main point. If $v_1 < v_2$ for good A, Theorem 1 shows that $\widetilde{q}_A^M(1|v_1, v_2) = 0$ and $\widetilde{q}_A^M(2|v_1, v_2) = 1$, where $\widetilde{q}_A^M(1|v_1, v_2) \equiv$

[31] In addition, the optimal allocation implies more than the independence from Proposition 1; it says that good A's allocation to buyer i depends neither on buyer i's valuation on the other good B, w_i, nor on the other buyers' valuations on the other good, w_{-i}.

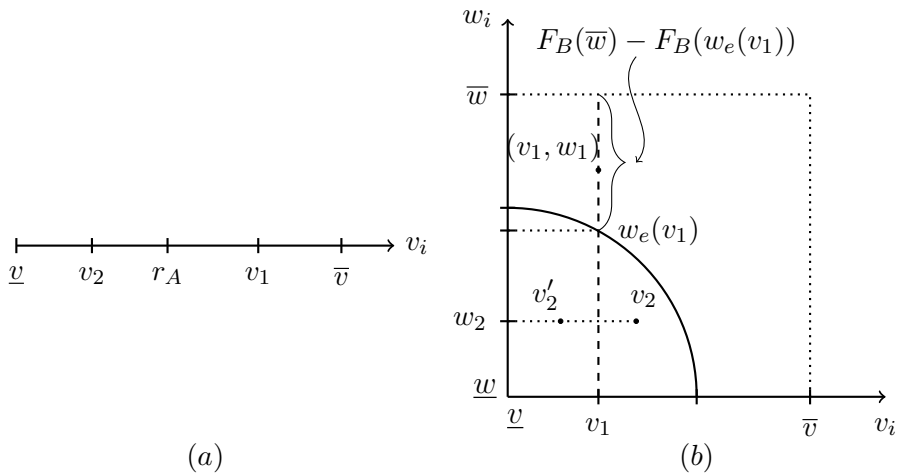

Figure 3: Single dimension (a) and Two dimensions (b)

$\int_{\underline{w}}^{\overline{w}} q_A^M(1|v_1, w_e(v_1), v_2, w_2) f_B(w_2) dw_2$ and $\widetilde{q}_A^M(2|v_1, v_2) \equiv \int_{\underline{w}}^{\overline{w}} q_A^M(2|v_1, w_1, v_2, w_e(v_2)) f_B(w_1) dw_1$ from (14). Hence, this entails

$$\int_{\underline{w}}^{\overline{w}} q_A^M(1|v_1, w_e(v_1), v_2, w_2) f_B(w_2) dw_2 = 0 \Rightarrow q_A^M(1|v_1, w_e(v_1), v_2, w_2) = 0 \text{ almost all } w_2 \in [\underline{w}, \overline{w}].$$

That is, even when buyer 2 is *not a member*, if $v_2 > v_1$, the optimal mechanism assigns zero allocation to buyer 1. Of course, in this way, the seller fails to accumulate bidder 2's payment when he is not a member – that is, neither buyer 1's payment nor buyer 2's. Still, in terms of the *expected* payment given v_2, it is optimal to do so. Specifically, for two bidders, the revenue in (22) can be rewritten in terms of $\widetilde{q}_A^M(1|v_1, v_2)$ and $\widetilde{q}_A^M(2|v_1, v_2)$ such as

$$\begin{aligned}
&\int_{\underline{v}}^{\overline{v}} Q_A^M(v_1)[1 - F_B(w_e(v_1))]\psi_A(v_1) f_A(v_1) dv_1 + \int_{\underline{v}}^{\overline{v}} Q_A^M(v_2)[1 - F_B(w_e(v_2))]\psi_A(v_2) f_A(v_2) dv_2 \\
= &\int_{\underline{v}}^{\overline{v}} \widetilde{q}_A^M(1|v_1, v_2)[1 - F_B(w_e(v_1))]\psi_A(v_1) dF_A(v_1) \times F_A(v_2) \\
&+ \int_{\underline{v}}^{\overline{v}} \widetilde{q}_A^M(2|v_1, v_2)[1 - F_B(w_e(v_2))]\psi_A(v_2) dF_A(v_1) \times F_A(v_2),
\end{aligned}$$

where $\widetilde{q}_A^M(1|v_1, v_2) + \widetilde{q}_A^M(2|v_1, v_2) \leq 1$. The formula, by the regularity and the monotonicity of $[1 - F_B(w_e(v_i))]$, shows that for $\widetilde{q}_A^M(2|v_1, v_2) = 1$ given $v_2 > v_1$, the seller can in fact collect bidder 2's payment only with probability $[1 - F_B(w_e(v_2))]$, *the probability that he is a member* given v_2. The origin of this new interpretation is based on the nature of the multi-dimensional problem. Unlike in a single dimension where no participation by bidder 2 implies $v_2 < v_1$ as in Figure 3 (a), in two dimensions, no participation by bidder 2 does not necessarily imply $v_2 < v_1$ as in Figure 3 (b).[32]

[32]That is, for a mechanism, even a non-member *reports*, so good A's allocation following the alternative

4.3 Separability & independence of payment

Now, we provide the separability and independence of a payment rule by examining the additional term $\int_{M(m)} u_A(v_e(w_i), w_i) dF_A(v_i) \times F_B(w_i)$ in (19). First, consider the interim payoff of a (member) buyer with a type (v_i, w_i) if he reports truthfully such that

$$u(v_i, w_i) = \int_{\underline{v}}^{v_i} Q_A^M(x)dx + \int_{\underline{w}}^{w_i} Q_B^M(y)dy + u_A(v_e(w_i), w_i) + u_B(v_i, w_e(v_i)) - e. \tag{23}$$

On the other hand, if the buyer misreports (v_i, w_i'), his payoff is

$$\int_{\underline{v}}^{v_i} Q_A^M(x)dx + w_i Q_B^M(w_i') - \left[w_i' Q_B^M(w_i') - \int_{\underline{w}}^{w_i'} Q_B^M(y)dy \right] + u_A(v_e(w_i'), w_i') + u_B(v_i, w_e(v_i)) - e. \tag{24}$$

For intuition behind corollary below, suppose that $u_A(v_e(w_i'), w_i')$ is differentiable (see the proof for when $u_A(v_e(w_i'), w_i')$ is not differentiable). Then, by taking the derivative of the payoff if buyer with (v_i, w_i) misreports (v_i, w_i') in (24), we have the first order condition:

$$\left[w_i - w_i'\right] \frac{dQ_B^M(w_i')}{dw_i'} - Q_B^M(w_i') + Q_B^M(w_i') + \frac{du_A(v_e(w_i'), w_i')}{dw_i'} = 0.$$

The differentiability of $Q_B^M(y) = G_B(y)$ from Theorem 1 hence implies that any incentive compatible mechanism (*i.e.*, $w_i' = w_i$) requires $u_A(v_e(w_i), w_i) = u_A(v_e(w_i'), w_i')$ for each pair $(v_i, w_i), (v_i, w_i') \in M(m)$ to satisfy the first order condition above. And this *constant* payment $u_A(v_e(w_i), w_i)$, not depending on w_i, can be transferred to a part of a membership fee without loss of generality.[33] As a result, the payment's independence and separability follow from (17).

Corollary 1 *Suppose H_k^M is regular for all $k \in \{A, B\}$. Among all **M**-mechanisms satisfying (16) and BC, those **M**-mechanisms maximizing the revenue make transfers for two goods separably and independently on $M(m)$ such that*

(i) $T_A^M(v_i, w_i) = T_A^M(v_i, w_i') = v_i Q_A^M(v_i) - \int_{\underline{v}}^{v_i} Q_A^M(x)dx$ for each pair $(v_i, w_i) \neq (v_i, w_i') \in M(m)$.

(ii) $T_B^M(v_i, w_i) = T_B^M(v_i', w_i) = w_i Q_B^M(w_i) - \int_{\underline{w}}^{w_i} Q_B^M(y)dy$ for each pair $(v_i, w_i) \neq (v_i', w_i) \in M(m)$.

SPA **M**-mechanism in Example 1, $q_A(1|\theta_1, \theta_2) = 0$ from $v_2 > v_1$, involves no problem when buyer 1 is only a bidder. However, for its implementation as an auction setting, a non-member simply does not make a bid, so to implement the same winning probability as the SPA **M**-mechanism, the canonical second price *auction* needs to be reconstructed with a random reserve price.

[33] To see it closely, from the interim payment for good A in (17), denote the constant $u_A(v_e(w_i), w_i)$ by $\overline{u}_A$ (resp., $\overline{u}_B$ for good B). Then, the interim payoff in (23) is given as $\int_{\underline{v}}^{v_i} Q_A^M(x)dx + \int_{\underline{w}}^{w_i} Q_B^M(y)dy + \overline{u}_A + \overline{u}_B - e$, where the seller or the mechanism designer can make a buyer pay an additional fixed amount $\overline{u}_A$ on top of $v_i Q_A^M(v_i) - \int_{\underline{v}}^{v_i} Q_A^M(x)dx$ and also from good B's sale $\overline{u}_B$. This is equivalent to a membership fee $e' = e - (\overline{u}_A + \overline{u}_B)$, without those fixed amounts.

Following the corollary, like $Q_k^M(v_i)$, we denote $T_A^M(v_i) = T_A^M(v_i, w_i) = T_A^M(v_i, w_i')$ for each $w_i \neq w_i'$ and $T_B^M(w_i) = T_B^M(v_i, w_i) = T_B^M(v_i', w_i)$ for each $v_i \neq v_i'$. Then, the theorem and its corollary result in the optimal payment rule $T_A(v_i) = \int_{\underline{v}}^{v_i} x dG_A(x)$ and $T_B(w_i) = \int_{\underline{w}}^{w_i} y dG_B(y)$.

One can check that the alternative SPA **M**-mechanism's allocation rule $\widetilde{q}_A^M$ in Example 1 satisfies Theorem 1 and its payment rule $\widetilde{t}_A^M$ satisfies Corollary 1. Yet, that payment rule is *one* possible rule to yield the interim payment from the Corollary; in fact, there is even a simpler rule. For the example, consider a different transfer rule $\widehat{t}_A^M$ such that

$$\widehat{t}_A^M(1|\theta_1, \theta_2) = \begin{cases} v_2 & \text{if } v_1 > v_2, \\ 0 & \text{if } v_1 < v_2. \end{cases}$$

To implement it as an auction, a natural institutional setting is the one with a random reserve price, which is discussed in Section 7.

4.4 Equivalence and optimal membership fee

In the previous subsections, unlike the standard procedure, we cannot characterize the seller's choice for finding the optimal mechanism; rather, we consider a larger choice set and find the optimal mechanism subject to it. That is, if an M-mechanism satisfies IC and IR via (16) and BC, the optimal $Q_A^M(v_i)$ and $Q_B^M(w_i)$ are given as $G_A(v_i)$ and $G_A(v_i)$ by Theorem 1 and $T_A^M(v_i)$ and $T_B^M(w_i)$ are determined by Corollary 1. What remains to show is then the sufficiency: Those allocation rules and payment rules are incentive compatible and individually rational via MIC and BC. First, showing MIC follows from the standard approach, *i.e.*, using (18) without $u_A(v_e(w_i), w_i) - u_A(v_e(w_i'), w_i')$ for good A. That is, $\int_{v_i}^{v_i'} Q_A^M(v_i')dx - \int_{v_i}^{v_i'} Q_A^M(x)dx \geq 0$ with the increasing $Q_A^M(v_i) = G_A(v_i)$, and similarly for good B

This inequality and that for good B can be restated such that for each $(v_i, w_i), (v_i, w_i'), (v_i', w_i) \in M(m)$,

$$v_i Q_A^M(v_i) - T_A^M(v_i) \geq v_i Q_A^M(v_i') - T_A^M(v_i'), \tag{25}$$

$$w_i Q_B^M(w_i) - T_B^M(w_i) \geq w_i Q_B^M(w_i') - T_B^M(w_i'), . \tag{26}$$

From this step, showing BC is straightforward: $Q_A^M(v_i)$ and $Q_B^M(w_i)$ are given as $G_A(v_i)$ and $G_A(v_i)$, with no $M(m)$ in them, and $T_A^M(v_i)$ and $T_B^M(w_i)$ are determined by such rules. For the revenue maximization, the way that the seller chooses a member set should not be different from the result of the optimal mechanism given a membership fee e. Then, BC holds if for each $\theta_i \in \overline{M}(m)$,

$$v_i Q_A^M(v_i) - T_A^M(v_i) + w_i Q_B^M(w_i) - T_B^M(w_i) - e = 0. \tag{27}$$

Since a member set is not "relevant" for the optimal allocation and transfer, an **M**-mechanism's boundary condition BC in (9) simply yields a member set. In essence, Theorem 1 *reduces*

the problem of choosing a member set and a membership fee of an **M**-mechanism into choosing only a membership fee. The two notorious problems with multidimensional analysis, integrability and arbitrary rent – the payoff of each dimension's lowest type in the present model – are suggested by Rochet and Choné (1998). Theorem 1 is instrumental in resolving them for a membership mechanism.

As shown, since for the optimal mechanism, the member criterion depends only on a membership fee e, the member set can be rewritten in terms of a fee e such as $M^*(e) \equiv \{\theta_i \in \Theta_i : v_i Q_A^M(v_i) - T_A^M(v_i) + w_i Q_B^M(w_i) - T_B^M(w_i) \geq e\}$, instead of $M(m)$ given the entire mapping m. By incorporating the optimal allocation and transfer rule, from Theorem 1 and Corollary 1, into the revenue in (22), (28) becomes the **M**-mechanism's maximum revenue from a single buyer *given* each membership fee e. Finally, the statement for the optimal mechanism is written in terms of the equivalence relationship, that is, among all incentive compatible and individually rational **M**-mechanisms – not a larger set satisfying (16) and BC – in the last result of this section below.

Corollary 2 *Suppose H_k^M is regular for all $k \in \{A, B\}$. Among all incentive compatible and individually rational **M**-mechanisms, those **M**-mechanisms maximizing the revenue choose the allocation by Theorem 1 and the payment by Corollary 1. Hence, those **M**-mechanisms maximizing the revenue choose a membership fee e such that*

$$\max_e \; R_M^*(e) \equiv \int_{M^*(e)} [T_A(v_i) + T_B(w_i) + e] \, dF_A(v_i) \times F_B(w_i), \tag{28}$$

where $T_A(v_i) = \int_{\underline{v}}^{v_i} x dG_A(x)$ and $T_B(w_i) = \int_{\underline{w}}^{w_i} y dG_B(y)$, and

$$M^*(e) = \left\{ \theta_i \in \Theta_i : \int_{\underline{v}}^{v_i} G_A(x)dx + \int_{\underline{w}}^{w_i} G_B(y)dy \geq e \right\}. \tag{29}$$

Furthermore, the mapping $\int_{\underline{v}}^{v_i} G_A(x)dx + \int_{\underline{w}}^{w_i} G_B(y)dy$ is continuous and monotonic in θ_i.

Since we start by assuming that $m(\cdot)$ in (1) is continuous and weakly monotonic, to complete the whole procedure, the mapping from $M^*(e)$ given a mechanism satisfying IC and IR through BC must satisfy the two properties. In addition, the existence of the optimal e, a solution to the constrained maximization problem (28), follows from the usual argument.

5 Dominance of membership

In this section, we provide the second main result of this paper, conditions under which membership dominates separate participation. The optimal membership mechanism from Section 4 is necessary for the proper comparison because we examine the dominance of separate participation as well as the dominance of membership. Technically, in addition, even if an **M**-mechanism dominates separate participation in a local sense, there might not exist a solution to the former problem. Hence, the previous section *enables* us to compare

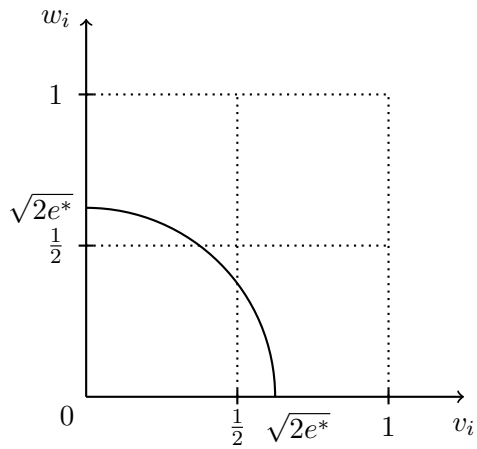

Figure 4: Dominance in the uniform case

them. We examine the difference in their revenues in terms of a single buyer; the total difference is then that difference multiplied by N.

The maximum membership revenue is from (28) in the previous section. On the other hand, the maximum revenue for separate participation is the duplication of Myerson (1981): with the two optimal reserve prices, r_A and r_B,

$$R_A^*(r_A) + R_B^*(r_B), \tag{30}$$

where

$$R_A^*(r_A) \equiv r_A(1 - F_A(r_A))G_A(r_A) + \int_{r_A}^{\overline{v}} \left[\int_{r_A}^{v_i} x dG_A(x) \right] dF_A(v_i), \tag{31}$$

and similarly, $R_B^*(r_B)$ can be found. As well known, for separate participation, given each good, the optimal revenue can be obtained from assigning each item to the highest valuation with ex-ante reserve prices.[34]

In general, finding a closed-form solution for the **M**-mechanism's revenue maximizing fee e in (28) is intractable. The complications arise from the multidimensional integral in the member set in (29) especially with more than two bidders in addition to general valuation distributions for two goods. Yet, the simple environment in Example 1 still enables us to do so, from which one can also catch a glimpse of the difficulties. We revisit Example 1 with a symmetric and uniform distribution, $F = F_A = F_B = U[0, 1]$.

Example 2 (two bidders, symmetric & uniform distribution) With the symmetric distribution, we suppress the subscripts k in F_k, H_k^M and T_k. From (29), the member set is

[34]By revenue equivalence, for example, in the second-price auction, a buyer pays r_A if all other buyers' valuations are smaller than r_A and pays the second highest valuation conditional on his valuation being the highest otherwise. The equivalence between a reserve price and an entry fee for such a role is also well known.

given as

$$M^*(e) = \left\{ (v_i, w_i) \in [0,1]^2 : \frac{1}{2}v_i^2 + \frac{1}{2}w_i^2 - e \geq 0 \right\}.$$

Using (22), the expected revenue from a buyer in (28) is $R_M^*(e) = \int_{M^*(e)}[T(v_i) + T(w_i) + e]dF(v_i) \times F(w_i) = 2\int_0^1 x\psi(x)h^M(x)dx + eH^M(1)$, which yields, by rewriting the integral over the member set as the integral over the whole feasible set minus that over the non-member set,

$$R_M^*(e) = 2\int_0^1 (2x^2 - x)dx + e - \int_0^{\sqrt{2e}} (x^2 + e)\sqrt{2e - x^2}dx = \int_0^1 (4x^2 - 2x)dx + e - \left(\frac{3\pi}{4}\right)e^2.$$

The optimal membership fee is then $e^* = \frac{2}{3\pi}$, and the maximum expected revenue from a buyer is $\int_0^1 (4x^2 - 2x)dx + \frac{1}{3\pi}$.[35] With the optimal fee, as describe in Figure 4, we find that $r = \frac{1}{2} < \sqrt{2e^*}$ and $\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 > 2e^*$, where $r = \frac{1}{2}$ is the uniform's optimal reserve price for separate participation. Hence, with membership, for a relatively small $w_i > 0$, the minimal qualification $v_e(w_i)$ is greater than the optimal reserve price, whereas for a relatively high w_i, it is lower than that. The maximum revenue for separate participation is $R_A^*(r) + R_B^*(r) = 2\int_{1/2}^1 (2z^2 - z)dz$. We compare them to show the dominance of membership such that

$$R_M^*(e^*) - [R_A^*(r) + R_B^*(r)] = \int_0^1 (4x^2 - 2x)dx + \frac{1}{3\pi} - \int_{1/2}^1 (4z^2 - 2z)dz = \int_0^{1/2} (4x^2 - 2x)dx + \frac{1}{3\pi} > 0.$$

The membership yields a higher revenue.

The approach in Example 2 is not applicable to general cases, including non-uniform distributions. Instead, we make *connections* between the two mechanisms, membership and separate participation, through the member standard line. Specifically, we do so by choosing a membership fee such that one of the member standard line's intercepts, $\widehat{v}_e$ or $\widehat{w}_e$, is identical to either the optimal reserve price for good A or that for good B of separate participation. To connect them, we substitute both "corner types" $(r_A, \underline{w})$ and $(\underline{v}, r_B)$ into the member standard line $\int_{\underline{v}}^{v_i} G_A(x)dx + \int_{\underline{w}}^{w_i} G_B(y)dy = e$ from (29). Suppose, without loss of generality, $\int_{\underline{v}}^{r_A} G_A(x)dx \geq \int_{\underline{w}}^{r_B} G_B(y)dy$, as found from Figure 5 (a).[36] Then, we choose a membership fee e to make the member standard line's vertical intercept the same as good B's optimal reserve price; that is, we choose e such that $e = \widehat{e}_N$, where we denote

$$\widehat{e}_N \equiv \int_{\underline{w}}^{r_B} G_B(y)dy. \tag{32}$$

We include the number of buyers N above to keep track of its changes with respect to changes in N explicitly. Then, by choosing $e = \widehat{e}_N$, the membership's maximum revenue given $\widehat{e}_N$ –

[35] Obviously, the revenue in terms of the virtual valuation, $\int_0^1 (2x^2 - x)dx$, is identical to the revenue without the reformulation, $\int_0^1 T(x)dx$, where $T(x) = \frac{1}{2}x^2$.

[36] Note that, unlike the uniform example in Figure 4, *general* member standard lines in Figure 5 are not necessarily circular; it can be any decreasing function.

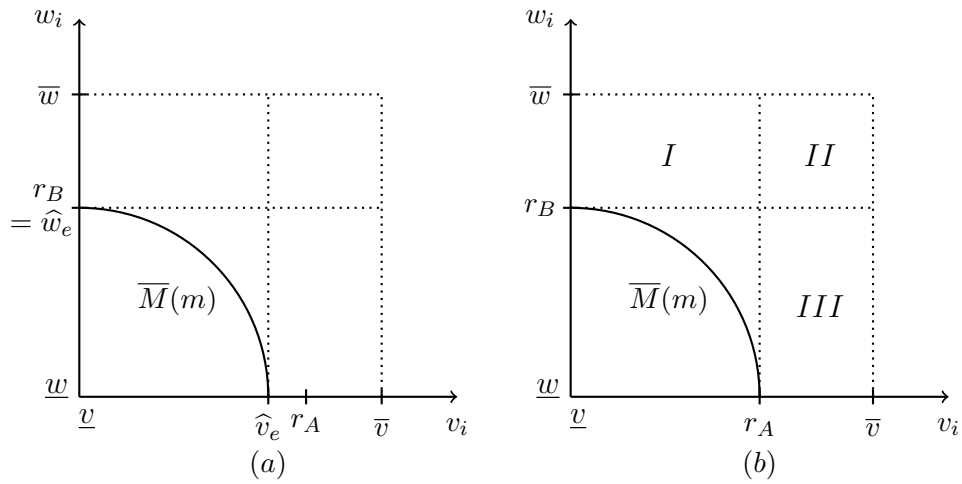

Figure 5: Connection between two mechanisms (a) and symmetric case (b)

not necessarily an optimal fee – in (28) yields good A's payment from a buyer such that

$$\int_{M^*(\widehat{e}_N)} [T_A(v_i)]\, dF_A(v_i) \times F_B(w_i) \tag{33}$$
$$= \int_{\underline{v}}^{r_A} \int_{w_e(v_i)}^{\overline{w}} \left[\int_{\underline{v}}^{v_i} x dG_A(x) \right] dF_B(w_i) dF_A(v_i) + \int_{r_A}^{\overline{v}} \left[\int_{\underline{v}}^{v_i} x dG_A(x) \right] dF_A(v_i).$$

The first term aggregates good A's valuations of buyer types who become members even when their valuations for good A are lower than the separate participation's reserve price r_A, and the second term aggregates good A's valuations of member types whose valuations for good A are greater than that.[37] Unlike separate participation, for membership, expanding the participating type set being positive, the seller cannot prevent buyers from bidding low in the absence of an ex-ante reserve price, *i.e.*, $\int_{\underline{v}}^{v_i} x dG_A(x)$ (in Section 7, no ex-ante reserve price changes to a *lower* ex-ante reserve price for membership than the optimal reserve price for separate participation).[38] The trade-off becomes apparent from comparison with separate participation's revenue.

The difference in good A's payment between membership (33) given $\widehat{e}_N$ and the optimal

[37]We can use (33) for both the general case and the symmetric case since the same formula is derived regardless of whether the member standard line's horizontal intercept is the same as the optimal reserve price for good A or not, that is, $\int_{\underline{v}}^{r_A} G_A(x)dx = \int_{\underline{w}}^{r_B} G_B(y)dy$ or $\int_{\underline{v}}^{r_A} G_A(x)dx > \int_{\underline{w}}^{r_B} G_B(y)dy$; in other words, regardless of $\widehat{v}_e < r_A$ in Figure 5 (a) or $\widehat{v}_e = r_A$ in Figure 5 (b).

[38]Specifically, in membership with ex-ante reserve prices or absolute exclusion in Section 7, a fixed reserve price is still *lower* than separate participation's reserve price r_A, so the same argument holds; the seller cannot prevent buyers from bidding low with a relatively lower reserve price than r_A.

separate participation (31), then, is derived as follows.[39]

$$\Delta R_A(r_A) \equiv \int_{\underline{v}}^{r_A} \left(1 - F_B(w_e(v_i))\right) \left[\int_{\underline{v}}^{v_i} x dG_A(x)\right] dF_A(v_i) + (1 - F_A(r_A)) \int_{\underline{v}}^{r_A} x dG_A(x) \tag{34}$$
$$- (1 - F_A(r_A)) r_A G_A(r_A).$$

The first two terms represent revenue gain from a buyer's payment for good A with membership, and the last term is revenue loss from no reserve price aggregation that could otherwise be accumulated with separate participation. Now, one may argue about the whole point of this analysis: Since for a fixed dimension A, this payment net effect is always negative from the optimal mechanism by Myerson (1981) – that is, with probability 1, it is optimal *not to assign* good A to buyers with valuations lower than r_A.[40] However, with membership, the seller can collect membership fees from those participating types. In other words, in return for the low bids negatively affecting the seller compared with separate participation, the expanded participating set provides the additional revenue. Thus, the importance of $\Delta R_A(r_A)$ is rather its role to provide a *lower bound* for such a loss, as will be shown below.

This difference in (34) can be succinctly rewritten as

$$\Delta R_A(r_A) = \int_{\underline{v}}^{r_A} \left(1 - F_B(w_e(v_i))\right) \left[\int_{\underline{v}}^{v_i} x dG_A(x)\right] dF_A(v_i) - (1 - F_A(r_A)) \int_{\underline{v}}^{r_A} G_A(x) dx, \tag{35}$$

where the second term is from $\int_{\underline{v}}^{r_A} G_A(x)dx = r_A G_A(r_A) - \int_{\underline{v}}^{r_A} x dG_A(x)$. We use (35) as a key formula to compare the revenue from membership with that from separate participation. Similarly, the difference in good B's payment between membership given $\widehat{e}_N$ and the optimal separate participation $\Delta R_B(r_B)$ can be found. Then, the total difference in revenue between membership with fee $\widehat{e}_N$ from (32) and the optimal separate participation is written, in terms of $\Delta R_A(r_A)$ in (35) and $\Delta R_B(r_B)$, as

$$R_M^*(\widehat{e}_N) - [R_A^*(r_A) + R_B^*(r_B)] = \Delta R_A(r_A) + \Delta R_B(r_B) + \widehat{e}_N H_A^M(\overline{v}),$$

where $\widehat{e}_N H_A^M(\overline{v})$ is the expected membership fee.

---

[39] Note that by rewriting (33), we have $\int_{\underline{v}}^{r_A} \left(1 - F_B(w_e(v_i))\right) \left[\int_{\underline{v}}^{v_i} x dG_A(x)\right] dF_A(v_i) + \int_{r_A}^{\overline{v}} \left[\int_{r_A}^{v_i} x dG_A(x)\right] dF_A(v_i) + \int_{r_A}^{\overline{v}} \left[\int_{\underline{v}}^{r_A} x dG_A(x)\right] dF_A(v_i)$, which yields (34).

[40] That is, with no ex-ante reserve price, (31) changes to $R_A(\underline{v}) = \int_{\underline{v}}^{\overline{v}} \left[\int_{\underline{v}}^{v_i} x dG_A(x)\right] dF_A(v_i)$, and for $e > 0$, the revenue for good A from membership (33) is lower than that from separate participation (31) such that

$$\int_{\underline{v}}^{r_A} (1 - F_B(w_e(v_i))) \left[\int_{\underline{v}}^{v_i} x dG_A(x)\right] dF_A(v_i) + \int_{r_A}^{\overline{v}} \left[\int_{\underline{v}}^{v_i} x dG_A(x)\right] dF_A(v_i) < R_A(\underline{v}) \leq R_A^*(r_A).$$

5.1 Symmetric case

We first study the case in which, given the reserve prices from separate participation, r_A and r_B, $\int_{\underline{v}}^{r_A} G_A(x)dx = \int_{\underline{w}}^{r_B} G_B(y)dy$ for all N as in Figure 5 (b). We call it a symmetric membership case, and clearly, a symmetric distribution, $F_A = F_B$, implies this symmetric case, but it can hold without such symmetry of the distribution. From $1 - F_B(w_e(v_i)) \geq 1 - F_B(r_B)$ for $v_i \in [\underline{v}, r_A]$, $\Delta R_A(r_A)$ in (34) satisfies the inequality as follows.

$$\Delta R_A(r_A) \geq \int_{\underline{v}}^{r_A} \big(1 - F_B(r_B)\big) \left[\int_{\underline{v}}^{v_i} x dG_A(x)\right] dF_A(v_i) - (1 - F_A(r_A)) \int_{\underline{v}}^{r_A} G_A(x)dx. \quad (36)$$

In this way, we only utilize the area $I+II+III$ in Figure 5 (b) since the remaining area above the member standard line can be tiny for some parameter values; that is, it is impossible to reach a definite answer, contingent on that area. Using $\Delta R_A(r_A)$ (similarly, $\Delta R_B(r_B)$), we aim to show that the payment *net loss* from membership is bounded from below for a sufficiently large number of bidders.

The right-hand side of the above equation in (36) can be rewritten as

$$\int_{\underline{v}}^{r_A} G_A(x)dx\big[\big(1 - F_B(r_B)\big)D_A(N) + F_A(r_A)F_B(r_B) - 1\big], \quad (37)$$

by the integration by parts,[41] and, in addition, by defining a term:

$$D_A(N) \equiv \frac{r_A F_A(r_A)G_A(r_A) - \int_{\underline{v}}^{r_A} v_i g_A(v_i)F_A(v_i)dv_i}{\int_{\underline{v}}^{r_A} G_A(v_i)dv_i}. \quad (38)$$

The following limit result provides a critical step to show membership's dominance in Theorem 2 by establishing a lower bound, despite complications in comparing the two different mechanisms.

Lemma 3 *For each $k \in \{A, B\}$, $\lim_{N\to\infty} D_k(N) \geq 1$.*

Hence, with the bound, the value in the bracket $[(1-F_B(r_B))D_A(N)+F_A(r_A)F_B(r_B)-1]$ of (37) is greater than equal to $-[1 - F_A(r_A)]F_B(r_B)$. A similar procedure can be applied to good B to obtain

$$\int_{\underline{w}}^{r_B} G_B(y)dy\,\big[\big(1 - F_A(r_A)\big)D_B(N) + F_A(r_A)F_B(r_B) - 1\big]\,, \quad (39)$$

where $\widehat{e}_N = \int_{\underline{v}}^{r_A} G_A(x)dx = \int_{\underline{w}}^{r_B} G_B(y)dy$. Additionally, the seller obtains membership fees as a part of the revenue, which is, in expectation, *at least* $[1 - F_A(r_A)F_B(r_B)]\widehat{e}_N$ from $I + II + III$ in Figure 5 (b). The total of the three revenue sources, with the limit result from Lemma 3, establishes that with the symmetric case, for a sufficiently large number of buyers, the revenue from membership is greater than that from separate participation.

[41] Note that $\int_{\underline{v}}^{r_A} \left[\int_{\underline{v}}^{v_i} x dG_A(x)\right] dF_A(v_i) = F_A(r_A)\left[r_A G_A(r_A) - \int_{\underline{v}}^{r_A} G_A(x)dx\right] - \int_{\underline{v}}^{r_A} v_i g_A(v_i)F_A(v_i)dv_i$.

Theorem 2 *Suppose H_k^M is regular for all $k \in \{A, B\}$. Consider the symmetric case. There exists $\overline{N}$ such that if $N \geq \overline{N}$, membership dominates separate participation.*

To understand intuition behind the Theorem, the probability $[1 - F_A(r_A)F_B(r_B)]$ for collecting membership fees can be rewritten as

$$1 - F_A(r_A)F_B(r_B) = (1 - F_B(r_B))F_A(r_A) + (1 - F_A(r_A))F_B(r_B) + (1 - F_A(r_A))(1 - F_B(r_B)),$$

with which we decompose (36) and the same term for good B in such a way that the net benefit from membership given each dimension includes *membership fees* for valuations lower than a corresponding reserve price. That is, in light of the three areas in Figure 5 (b), we have

$$\underbrace{\int_{\underline{v}}^{r_A} \big(1 - F_B(r_B)\big) \left[\int_{\underline{v}}^{v_i} x dG_A(x)\right] dF_A(v_i)}_{\text{Payment gain for } A \text{ from } I} + \underbrace{(1 - F_B(r_B))F_A(r_A)\widehat{e}_N}_{\text{Membership fee from } I} - \underbrace{(1 - F_A(r_A)) \int_{\underline{v}}^{r_A} G_A(x)dx}_{\text{Payment loss for } A \text{ from } II+III},$$

and

$$\underbrace{\int_{\underline{w}}^{r_B} \big(1 - F_A(r_A)\big) \left[\int_{\underline{w}}^{w_i} y dG_B(y)\right] dF_B(w_i)}_{\text{Payment gain for } B \text{ from } III} + \underbrace{(1 - F_A(r_A))F_B(r_B)\widehat{e}_N}_{\text{Membership fee from } III} - \underbrace{(1 - F_B(r_B)) \int_{\underline{w}}^{r_B} G_B(y)dy}_{\text{Payment loss for } B \text{ from } I+II},$$

and the expected membership fee from II, which is

$$\underbrace{(1 - F_A(r_A))(1 - F_B(r_B))\widehat{e}_N.}_{\text{Membership fee from } II}$$

Consider the *lowest* bound from the limit in Lemma 3 for both good A and B, $\lim_{N\to\infty} D_k(N) = 1$. Then, the total difference converges as follows.

$$\begin{cases} \widehat{e}_N \left[\big(1 - F_B(r_B)\big)D_A(N) + F_A(r_A)F_B(r_B) - 1\right] + (1 - F_B(r_B))F_A(r_A)\widehat{e}_N \\ +\widehat{e}_N \left[\big(1 - F_A(r_A)\big)D_B(N) + F_A(r_A)F_B(r_B) - 1\right] + (1 - F_A(r_A))F_B(r_B)\widehat{e}_N \\ +(1 - F_A(r_A))(1 - F_B(r_B))\widehat{e}_N. \end{cases}$$

$$\rightarrow \begin{cases} \widehat{e}_N \left[-F_B(r_B) + F_A(r_A)F_B(r_B)\right] + (1 - F_B(r_B))F_A(r_A)\widehat{e}_N \\ +\widehat{e}_N \left[-F_A(r_A) + F_A(r_A)F_B(r_B)\right] + (1 - F_A(r_A))F_B(r_B)\widehat{e}_N \\ +(1 - F_A(r_A))(1 - F_B(r_B))\widehat{e}_N. \end{cases}$$

One can find that the expected membership fee from I cancels out the net payment (loss) from good B, and the expected membership fee from III cancels out the net payment (loss) from good A. Those *cross-cancellations* reduce it into only the positive expected membership fee from II.

With a sufficiently large number of buyers, the positive effects of the additional revenue from membership dominate the negative effects of the low bids without a reserve price (or a lower reserve price in terms of the extended model in Section 7). Note that $R_M^*(e) -$

$[R_A^*(r_A) + R_B^*(r_B)]$ is the revenue difference from a *single* buyer, so the total difference the seller obtains from the comparison is $N \times \{R_M^*(e) - [R_A^*(r_A) + R_B^*(r_B)]\}$. The following example provides a more concrete idea on the comparison of N buyers.

Example 3 (N bidders, symmetric & uniform distribution) With the symmetric distribution, we suppress the subscripts k in F_k, G_k and D_k. Then, the difference in the expected revenues from a buyer is

$$R_M^*(e) - [R_A^*(r_A) + R_B^*(r_B)] \geq 2\int_0^r G(y)dy\left[\big(1 - F(r)\big)D(N) + F(r)F(r) - 1\right] + [1 - F(r)F(r)]\widehat{e}_N,$$

where $G(x) = x^{N-1}$ for $N = 2, 3, ...$ and $\widehat{e}_N = \int_0^r G(x)dx = \frac{1}{N}r^N$. Recall that the optimal reserve price with the uniform case for separate participation is $r = \frac{1}{2}$. Then, $D(N)$ in (38) is derived as

$$D(N) = \frac{rF(r)G(r) - \int_0^r vg(v)F(v)dv}{\int_0^r G(v)dv} = rN\left(1 - \frac{N-1}{N+1}\right) = \frac{2Nr}{N+1},$$

where we leave r as it is to see how the formula changes with the notation. Hence,

$$2\int_0^r G(x)dx\left[(1-r)\frac{2Nr}{N+1} + r^2 - 1\right] + (1 - r^2)\int_0^r G_A(x)dx = (1-r)\frac{r^N}{N}\left[\frac{4Nr}{N+1} - 1 - r\right],$$

which leads to

$$N\big\{R_M^*(e) - [R_A^*(r_A) + R_B^*(r_B)]\big\} \geq (1-r)r^N\left[\frac{4Nr}{N+1} - 1 - r\right].$$

The example is useful to make two points. One is that the result of Theorem 2 is based on a lower bound given the inequality (36). That is, although we can find that the above formula $(1-r)r^N\big[4Nr/(N+1) - 1 - r\big]$ rises first and then diminishes as N increases, the changes are for the lower bound of $N\big\{R_M^*(e) - [R_A^*(r_A) + R_B^*(r_B)]\big\}$, not for the actual difference itself. The other is that a sufficiently large N in Theorem 2 is a *sufficient* condition. With $r = \frac{1}{2}$, one can find that if $N = 2$, the above lower bound is *negative*. Yet, Example 2 shows that for $N = 2$, the closed-form solution for membership dominates separate participation.

5.2 General case

Even if $\int_{\underline{v}}^{r_A} G_A(x)dx > \int_{\underline{w}}^{r_B} G_B(y)dy$, as in Figure 5 (a), the same procedure from the symmetric case applies until the last step. There, now, with $\widehat{e}_N = \int_{\underline{w}}^{r_B} G_B(y)dy$, a formula different from (37) is derived as

$$\begin{aligned}&\int_{\underline{v}}^{r_A} G_A(x)dx\left[\big(1 - F_B(r_B)\big)D_A(N) + F_A(r_A)F_B(r_B) - 1\right] \\ &+ \widehat{e}_N\left[\big(1 - F_A(r_A)\big)D_B(N) + F_A(r_A)F_B(r_B) - 1\right] + [1 - F_A(\widehat{v}_e)F_B(r_B)]\widehat{e}_N.\end{aligned} \tag{40}$$

Then, if $\lim_{N\to\infty} D_k(N)$ in Lemma 3 is sufficiently close to 1, a negative value for $\big(1 - F_B(r_B)\big)D_A(N)+F_A(r_A)F_B(r_B)-1$, coupled with the inequality $\int_{\underline{v}}^{r_A} G_A(x)dx > \int_{\underline{w}}^{r_B} G_B(y)dy$, may hinder further progress to show the dominance.[42] Nevertheless, the dominance can be established if the following condition holds:

$$\frac{\int_{\underline{v}}^{r_A} G_A(x)dx}{\int_{\underline{w}}^{r_B} G_B(y)dy} \leq \frac{F_A(\widehat{v}_e)F_B(r_B) - 1}{F_A(r_A)F_B(r_B) - 1}. \tag{41}$$

It can be shown that for a sufficiently large N, $\int_{\underline{v}}^{r_A} G_A(x)dx$ and $\int_{\underline{w}}^{r_B} G_B(y)dy$ become sufficiently close to each other, so the above condition (41) is satisfied.

Theorem 3 *Suppose H_k^M is regular for all $k \in \{A, B\}$. There exists $\overline{N}'$ such that if $N \geq \overline{N}'$, membership dominates separate participation.*

The assumption (41) may require a stronger condition on the number of buyers, so $\overline{N}'$ from this asymmetric case can be different from $\overline{N}$ from the symmetric case of Theorem 2.

6 Dominance of separate participation

In this section, for an *optimal* membership fee e, we show the dominance of separate participation. To show it, we choose a reserve price for good A and a reserve price for good B such that each reserve price is the same as the corresponding good's intercept from e of membership; $r'_A = \widehat{v}_e$ and $r'_B = \widehat{w}_e$. That is, the freedom to choose two variables separately enables us to examine the general distribution case with two different intercepts, unlike the dominance of membership – there is no need to divide the analysis into the symmetric case and the general case.

We incorporate the degree of first-order stochastic dominance by *parameterizing* any given distribution $F_A(v_i)$ or $F_B(w_i)$ with $n \in \mathbb{N}$ such as $F_A(v_i)^n$ and $F_B(w_i)^n$. Then, the key formula in (35) for good A is modified with the parameterization of the stochastic dominance as below.

$$\int_{\underline{v}}^{r'_A} \big(1 - F_B(w_e(v_i))^n\big) \left[\int_{\underline{v}}^{v_i} x dG_A(x)\right] dF_A(v_i)^n - (1 - F_A(r'_A)^n) \int_{\underline{v}}^{r'_A} G_A(x)dx, \tag{42}$$

where with n, the winning probabilities for both goods in Theorem 1 can be rewritten as $G_A(x) = F_A(x)^{n(N-1)}$ and $G_B(y) = F_B(y)^{n(N-1)}$. From $1 - F_B(w_e(v_i)) \leq 1$ for $v_i \in [\underline{v}, r_A]$, the key formula (35) satisfies inequality:

$$\Delta R_A(r'_A) \leq \int_{\underline{v}}^{r'_A} \left[\int_{\underline{v}}^{v_i} x dG_A(x)\right] dF_A(v_i)^n - (1 - F_A(r'_A)^n) \int_{\underline{v}}^{r'_A} G_A(x)dx. \tag{43}$$

[42] That is, if $\lim_{N\to\infty} D_k(N)$ in Lemma 3 is sufficiently close to 1, $\big(1 - F_B(r_B)\big)D_A(N) + F_A(r_A)F_B(r_B) - 1 < 0$ and $\int_{\underline{v}}^{r_A} G_A(x)dx > \int_{\underline{w}}^{r_B} G_B(y)dy = \widehat{e}_N$, so the following reversed inequality impedes the next step: $\int_{\underline{v}}^{r_A} G_A(x)dx \left[\big(1 - F_B(r_B)\big)D_A(N) + F_A(r_A)F_B(r_B) - 1\right] < \widehat{e}_N \left[\big(1 - F_B(r_B)\big)D_A(N) + F_A(r_A)F_B(r_B) - 1\right]$.

Then, as shown in the proof of the following theorem, for the *reverse* dominance, we examine only the following two terms out of the right-hand side of (43) such that

$$r'_A G_A(r'_A) F_A(r'_A)^n - (1 - F_A(r'_A)^n) \int_{\underline{v}}^{r'_A} G_A(x)dx, \tag{44}$$

and, similarly, for good B, we only need to consider $r'_B G_B(r'_B) F_B(r'_B)^n - (1 - F_B(r'_B)^n) \int_{\underline{w}}^{r'_B} G_B(y)dy$. The theorem below establishes that for a sufficiently large n – *i.e.*, more stochastic dominant distributions – the revenue from separate participation is greater than that from membership.

Theorem 4 *Suppose H_k^M is regular for all $k \in \{A, B\}$. There exists $\overline{n}$ such that if $n \geq \overline{n}$, separate participation dominates membership.*

If the valuation distributions become sufficiently stochastic dominant, the negative effects of the low bids without a fixed reserve price outweigh the positive effects of the additional revenue from membership fees.

7 Extensions

7.1 Implementation

The increasing modified virtual valuation results in the simple optimal allocation rule for the **M**-mechanism in Theorem 1, but its implementation as an auction may not be simple. For a mechanism, even those non-members report their types *for zero allocation*, but for an auction, they are not simply bidding. Then, implementing the allocation of the alternative mechanism in Example 1, optimal from Theorem 1, is not possible, since to do so requires bidder 1's winning to depend on bidder 2's valuation even if the latter is not a member.

Yet, the challenge opens up a new and interesting area with respect to its implementation. We construct a second price auction for the optimal mechanism by introducing a random reserve price to implement the optimal allocation as well as the payment rule. To delve into the issue further, consider a two-bidder case. Suppose that the seller adopts a second price auction now given bidders' bids (b_1, b_2), not their type reports. Then, for good A, bidder i's payoff from bidding b_i is

$$\begin{cases} v_i - b_j & \text{if } b_i > b_j, \\ 0 & \text{if } b_i < b_j, \end{cases}$$

and the standard tie-breaking applies. Hence, given a symmetric and monotonic bidding strategy β_A, bidder i's winning probability is $\Pr(\theta_2 \notin M(m)) + \Pr\big(\theta_2 \in M(m), b_1 > \beta_A(V_2))\big)$, where the former is the probability that bidder 2 is not a member – so he does not make a bid, like the canonical SPA **M**-mechanism from Example 1. Then, his expected payoff is given as

$$\begin{aligned} &[\Pr(\theta_2 \notin M(m)) + \Pr\big(\theta_2 \in M(m), b_1 > \beta_A(V_2)\big)]v_i \\ &- \Pr\big(\theta_2 \in M(m), b_1 > \beta_A(V_2)\big)\mathbb{E}\left[\beta_A(V_2) \mid \theta_2 \in M(m), b_1 > \beta_A(V_2)\right], \end{aligned}$$

where the second term, bidder i's expected payment, is $\int_{\underline{v}}^{\beta_A^{-1}(b_i)} \int_{w_e(x)}^{\overline{w}} \beta_A(x) f_B(y) f_A(x) dy dx$. One can find that each bidder's Bayesian equilibrium strategy is still $\beta_A(v_i) = v_i$. However, in this way, bidder 1's winning probability does not depend on how he makes a bid if the other bidder does not participate, so this allocation rule is not optimal – different from the optimal allocation in Theorem 1 – such as

$$\Pr(\theta_2 \notin M(m)) + \Pr\left(\theta_2 \in M(m), v_1 > V_2\right) \neq F_A(v_1),$$

where $F_A(v_1)$ is the winning probability from the optimal allocation of the Theorem.

To implement the optimal winning probability as the auction counterpart of the optimal direct mechanism such as the alternative SPA **M**-mechanism in Example 1, the auction can be modified with a random reserve price such that

$$\begin{cases} v_i - b_j & \text{if } b_i > b_j \text{ given that bidder } j \text{ participates,} \\ v_i - \rho & \text{if } b_i > \rho \text{ given that bidder } j \text{ does not participate,} \\ 0 & \text{otherwise,} \end{cases}$$

where ρ denotes a random reserve price drawn according to the exact same valuation distribution for good A of bidder 2, that is, F_A. This can be an interesting and promising area for how an econometric analysis can be *practical*, beyond academic interest, in terms of its impact on actual allocation. That is, an econometrician can estimate a valuation distribution, and an auctioneer may use it to generate a random reserve price as above.

7.2 Ex-ante reserve price

Even with ex-ante reserve prices or absolute exclusion, if H_k^M for all $k \in \{A, B\}$ is regular, Theorem 1 empowers us to consider the simple optimal allocation such that each buyer's interim payment for the optimal **M**-mechanism given ex-ante reserve prices is given as

$$T_A(v_i, \delta_A) = G_A(\delta_A)\delta_A + \int_{\delta_A}^{v_i} x dG_A(x), \text{ and } T_B(w_i, \delta_B) = G_B(\delta_B)\delta_B + \int_{\delta_B}^{w_i} y dG_B(y),$$

where an ex-ante reserve price for good k is denoted by δ_k. With them and $\delta = (\delta_A, \delta_B)$, an **M**-mechanism's revenue maximization problem, equivalent to (28), is written as

$$\begin{aligned} \max_{e,\delta} \; R_M^*(e, \delta) \equiv & \int_{[\underline{v}, \delta_A] \times [\widehat{w}_{(e,\delta_B)}, \overline{w}]} \left[T_B(w_i, \delta_B) + e\right] dF_A(v_i) \times F_B(w_i) \\ & + \int_{[\widehat{v}_{(e,\delta_A)}, \overline{v}] \times [\underline{w}, \delta_B]} \left[T_A(v_i, \delta_A) + e\right] dF_A(v_i) \times F_B(w_i) \\ & + \int_{M^*(e,\delta)} \left[T_A(v_i, \delta_A) + T_B(w_i, \delta_B) + e\right] dF_A(v_i) \times F_B(w_i), \end{aligned}$$

where $\widehat{v}_{(e,\delta_A)}$ is given by $\int_{\delta_A}^{\widehat{v}_{(e,\delta_A)}} G_A(x)dx = e$; $\widehat{w}_{(e,\delta_B)}$ by $\int_{\delta_B}^{\widehat{w}_{(e,\delta_B)}} G_B(y)dy = e$; and

$$M^*(e, \delta) = \left\{\theta_i \in [\delta_A, \overline{v}] \times [\delta_B, \overline{w}] : \int_{\delta_A}^{v_i} G_A(x)dx + \int_{\delta_B}^{w_i} G_B(y)dy \geq e\right\}.$$

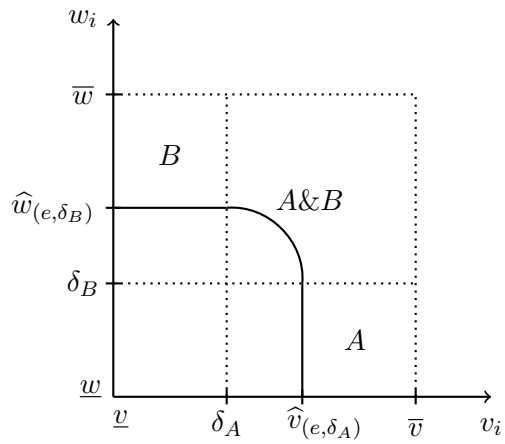

Figure 6: The case with two ex-ante reserve prices δ_A and δ_B

Unlike $M^*(e)$ in (29), this member set, as in Figure 6, consists of three areas: A (B) is a set of buyer types *only* bidding for good A (B), whereas $A\&B$ a set of types bidding both goods separately - note that the area is *not* for a bundle of them.

Despite the seemingly complex formulations, desirably, a similar but slightly more involved key formula can be obtained, corresponding to (35) for good A (similarly for good B). One consequence of having three variables $e, \delta = (\delta_A, \delta_B)$ is that given the optimal reserve prices r_A, r_B for separate participation, we can always have a symmetric case in the sense that by choosing $\delta, \widehat{e}$, $\int_{\delta_A}^{r_A} G_A(x)dx = \int_{\delta_B}^{r_B} G_B(y)dy = \widehat{e}$. Note that an ex-ante reserve price from this full-fledged membership model given $\widehat{e} > 0$ – that is, the *effective* membership case – is lower than the optimal reserve price from separate participation, that is, $\delta_k < r_k$ for all k. Then, a buyer's payment for good A, corresponding to (33), is derived as

$$\int_{\delta_A}^{r_A} \int_{w_{(e,\delta)}(x)}^{\overline{w}} \left[G_A(\delta_A)\delta_A + \int_{\delta_A}^{v_i} x dG_A(x) \right] dF_B(w_i) \times F_A(v_i) + \int_{r_A}^{\overline{v}} \left[G_A(\delta_A)\delta_A + \int_{\delta_A}^{v_i} x dG_A(x) \right] dF_A(v_i),$$

which yields the key formula in this case, corresponding to (35), such that

$$\Delta R_A(r_A, \delta) = \int_{\delta_A}^{r_A} \int_{w_{(e,\delta)}(x)}^{\overline{w}} \left[\delta_A G_A(\delta_A) + \int_{\delta_A}^{v_i} x dG_A(x) \right] dF_B(w_i) \times F_A(v_i) - (1 - F_A(r_A)) \int_{\delta_A}^{r_A} G_A(x)dx,$$

where $w_{(e,\delta_B)}(v_i) \equiv \min\{w_i \in [\delta_B, \overline{w}] : \theta_i \in M^*(e, \delta)\}$. Even without this formula, the dominance of membership with ex-ante reserve prices or absolute exclusion is implied by the dominance of pure membership, a membership with no absolute exclusion in Theorems 2 and 3, which is less interesting.

Yet, the dominance of separate participation over the effective membership given $\widehat{e} > 0$ with ex-ante reserve prices requires careful treatment. As in subsection 5.1., we incorporate the degree of first-order stochastic dominance by parameterizing any given distribution $F_A(v_i)$ or $F_B(w_i)$ with $n \in \mathbb{N}$ such as $F_A(v_i)^n$ and $F_B(w_i)^n$, and given the optimal membership e, δ, now we choose r'_A and r'_B such that $\int_{\delta_A}^{r'_A} G_A(x)dx = e$ and $\int_{\delta_B}^{r'_B} G_B(y)dy = e$. Then,

the proof of Corollary 3 shows

$$\Delta R_A(r'_A, \delta) \leq r'_A G_A(r'_A) F_A(r'_A)^n - (1 - F_A(r'_A)^n) \int_{\delta_A}^{r'_A} G_A(x) dx, \tag{45}$$

which is a formula for an **M**-mechanism with ex-ante reserve prices that extends (44). The dominance of separate participation still holds.

Corollary 3 *Suppose H_k^M is regular for all $k \in \{A, B\}$. There exists $\overline{n}'$ such that if $n \geq \overline{n}'$, separate participation dominates membership with ex-ante reserve prices.*

7.3 Alternative interpretation

The model can also be interpreted as an auction house or platform with two different sellers in which each seller pays a fixed rate $\alpha \in (0, 1)$ of his item's final sale price. Then, the auction house collects the fees based on the fixed rate from the two different sellers and membership fees from buyers, which yields the auction house the revenue such that

$$\sum_{i \in I^M} \alpha[t_A^M(i|\theta) + t_B^M(i|\theta)] + \#I^M e = \alpha \Big\{ \sum_{i \in I^M} [t_A^M(i|\theta) + t_B^M(i|\theta)] + \#I^M e' \Big\}, \text{ where } e' \equiv \frac{e}{\alpha}.$$

Hence, for the interpretation of the model as an auction house or a platform with a fixed rate $\alpha \in (0, 1)$, the expected revenue is $\int_{M(m)} [\alpha T_A(\theta_i) + \alpha T_B(\theta_i) + e] \, dF_A(v_i) \times F_B(w_i)$, which can be rewritten as $\alpha \int_{M(m)} [T_A(\theta_i) + T_B(\theta_i) + e'] \, dF_A(v_i) \times F_B(w_i)$, where $e' \equiv \frac{e}{\alpha}$.

This expected revenue is identical to the expected revenue from the model in (11), except for the multiplication of the fixed rate α. Put differently, if the interests of the two parties, the auction house and both seller A and seller B, are perfectly aligned, there is no loss of generality, even if we solve the platform's problem with two sellers as a single mechanism designer problem with two goods.

8 Concluding remarks

This paper introduces a membership mechanism. The optimal allocation requires the monotonicity of a modified virtual valuation distribution reflecting a member set, and it results in the simple allocation rule resembling the single dimension's optimal allocation. Such a simple allocation settles a member set's existence problem in this multi-dimensional problem with membership. More importantly, the optimal mechanism can be a powerful justification for secret reserve prices, and further, by bundling present participation and future participation *in an auction*, such a scheme can be utilized for real-world auction mechanisms – for example, government procurement and auction design such as spectrum auctions by requiring membership instead of refundable fees.

We identify two main contrasting factors governing the dominance of membership over separate participation. One is the number of bidders and the other is the degree of the

stochastic dominance of the valuation distributions. One might argue that the membership's dominance can still be *utilized* for a monopoly screening problem for a single agent such that the seller assigns an item to a buyer with the same probability "imitating" the one from this paper with the large number of bidders, but in an auction, a buyer himself or herself engages in bidding as well; it would be far-fetched to call it a screening problem for a single agent in that case.

There are a few directions to extend this model. No attempt has been made to explicitly configure a collection of the two factors – the number of bidders and the stochastic dominance – that makes the seller indifferent. The multi-object aspect has been neglected by the two-sided market literature, and more interestingly – and in an equally challenging way – the simple characterization of this model can be the first step to enable the literature to consider two different sellers with nonidentical objects in a tractable model and find the optimal membership fee, suggesting yet another rationale for a platform: a "place" for buyers to participate in *multiple* sellers.

Appendix: Proofs

Proof of Lemma 1. If $\Theta_i \setminus M(m) = \emptyset$, the result trivially holds. Suppose $\Theta_i \setminus M(m) \neq \emptyset$. We establish, under (6), the equivalent relationship between (7) & (8) and (9). Show ($\Rightarrow$). Suppose (9) is not satisfied; there exists $\theta_i = (v_i, w_i) \in \overline{M}(m)$ such that $u(\theta_i) > 0$. Consider a ϵ-ball about θ_i, $B_\epsilon(\theta_i) \equiv \{x \in \mathbb{R}^2_+ : ||x - \theta_i|| < \epsilon\}$ for $\epsilon > 0$. First, given $\Theta_i \setminus M(m) \neq \emptyset$, there exists $\theta'_i = (v'_i, w'_i)$ sufficiently close to θ_i and $\theta'_i < \theta_i$. Formally, there exists a sufficiently small $\epsilon > 0$ such that $B_\epsilon(\theta_i) \cap \{x \in \mathbb{R}^2_+ : x < \theta_i\} \neq \emptyset$. Suppose, on the contrary, $B_\epsilon(\theta_i) \cap \{x \in \mathbb{R}^2_+ : x < \theta_i\} = \emptyset$, implying $\theta_i = (0, 0)$, which, together with the monotonicity of m, leads to $M(m) = \Theta_i$, contradicting $\Theta_i \setminus M(m) \neq \emptyset$. Then, choose $\theta'_i = (v'_i, w'_i) \in B_\epsilon(\theta_i) \cap \{x \in \mathbb{R}^2_+ : x < \theta_i\}$. Since θ'_i is not a member type, he obtains $0 < u(v_i, w_i)$, which is equivalently rewritten as

$$0 < v_i Q^M_A(v_i, w_i) + w_i Q^M_B(v_i, w_i) - T^M(v_i, w_i) - e.$$

Hence, for a sufficiently small $\epsilon > 0$,

$$0 < v'_i Q^M_A(v_i, w_i) + w'_i Q^M_B(v_i, w_i) - T^M(v_i, w_i) - e,$$

which violates the incentive compatibility (7). Now, show ($\Leftarrow$). Given that (9) and the monotonicity of m imply (8), it remains to show (7). Suppose (7) is not satisfied; there exists $(v_i, w_i) \in \Theta_i \setminus M(m)$ such that

$$0 < v_i Q^M_A(v'_i, w'_i) + w_i Q^M_B(v'_i, w'_i) - T^M(v'_i, w'_i) - e \text{ for some } (v'_i, w'_i) \in M(m).$$

But then, for each $(v''_i, w''_i) \in \overline{M}(m)$ such that $(v''_i, w''_i) > (v_i, w_i)$,

$$\begin{aligned} u(v''_i, w''_i) = 0 &< v_i Q^M_A(v'_i, w'_i) + w_i Q^M_B(v'_i, w'_i) - T^M(v'_i, w'_i) - e \\ &\leq v''_i Q^M_A(v'_i, w'_i) + w''_i Q^M_B(v'_i, w'_i) - T^M(v'_i, w'_i) - e, \end{aligned}$$

which contradicts the sum of the two inequalities in (6), so contradicts the incentive compatibility among members (6), even for $Q_A^M(v_i', w_i') = 0$ or $Q_B^M(v_i', w_i') = 0$ (or both). ■

Proof of Proposition 1. We show that for each incentive compatible mechanism on $M(m)$, the allocation for one good does not depend on the report of the other good's valuation. First, examine good A's incentive compatibility from (6):

$$v_i Q_A^M(\theta_i) - T_A^M(\theta_i) \geq v_i Q_A^M(v_i', w_i') - T_A^M(v_i', w_i'). \tag{46}$$

This is equivalent to $\forall\,(v_i, w_i) \neq (v_i', w_i') \in M(m)$,

$$v_i Q_A^M(\theta_i) - T_A^M(\theta_i) \geq v_i Q_A^M(v_i', w_i) - T_A^M(v_i', w_i), \tag{47}$$

$$v_i Q_A^M(\theta_i) - T_A^M(\theta_i) = v_i Q_A^M(v_i, w_i') - T_A^M(v_i, w_i'). \tag{48}$$

If we fix v_i or fix w_i, it is immediate that the incentive compatibility in (46) implies the incentive compatibility with (47)-(48). On the other hand, by combining (47) with (48), the latter implies the former. In particular, (48) shows that for each mechanism inducing each buyer to report w_i truthfully for a fixed v_i, the buyer should have an identical payoff.

From the incentive compatibility (47)-(48), for good A, a direct mechanism is incentive compatible on $M(m)$ if and only if Q_A^M is increasing in v_i; $\forall\,(v_i, w_i) \in M(m)$,

$$\begin{aligned} &T_A^M(v_i, w_i) \\ &= v_i Q_A^M(v_i, w_i) + \big(T_A^M(v_e(w_i), w_i) - v_e(w_i) Q_A^M(v_e(w_i), w_i)\big) - \int_{v_e(w_i)}^{v_i} Q_A^M(x, w_i)dx; \end{aligned} \tag{49}$$

and $\forall\,(v_i, w_i), (v_i, w_i') \in M(m)$,

$$v_i Q_A^M(v_i, w_i) - T_A^M(v_i, w_i) = v_i Q_A^M(v_i, w_i') - T_A^M(v_i, w_i').$$

The same procedure applies to good B's incentive compatibility. Now, consider $(v_i, w_i), (v_i, w_i') \in M(m)$ with $w_i \neq w_i'$. Define $D(v_i)$ such that

$$\begin{aligned} D(v_i) \equiv\; & v_e(w_i) Q_A^M(v_e(w_i), w_i) - T_A^M(v_e(w_i), w_i) + \int_{v_e(w_i)}^{v_i} Q_A^M(y, w_i)dy \\ & - \left[v_e(w_i') Q_A^M(v_e(w_i'), w_i') - T_A^M(v_e(w_i'), w_i') + \int_{v_e(w_i')}^{v_i} Q_A^M(y, w_i')dy \right]. \end{aligned}$$

Then, from (48) and (49), $D(v_i) = 0$ for all v_i. By the mean value theorem and the fundamental theorem of calculus, $D'(v_i) = 0$ for almost all v_i. ■

Proof of Lemma 2. Taking the derivative of the virtual valuation yields

$$\psi_A^{M\prime}(x) = 1 + \frac{[h_A^M(x)]^2 + h_A^{M\prime}(x)[H_A^M(\overline{v}) - H_A^M(x)]}{[h_A^M(x)]^2}$$

where

$$h_A^M(x) = \begin{cases} [1 - F_B(w_e(x))]f_A(x) & \text{if } x \le \widehat{v}_e, \\ f_A(x) & \text{if } x > \widehat{v}_e, \end{cases}$$

and

$$h_A^{M\prime}(x) = \begin{cases} -f_B(w_e(x))w_e'(x)f_A(x) + [1 - F_B(w_e(x))]f_A'(x) & \text{if } x \le \widehat{v}_e, \\ f_A'(x) & \text{if } x > \widehat{v}_e. \end{cases}$$

Since $w_e'(x) \le 0$, if F_A is convex, then $H_A(x)$ is regular. Suppose $F_A(x)$ is not convex. Examine the two ranges in (20) separately. First, if $x > \widehat{v}_e$, the modified virtual valuation distribution becomes the standard virtual valuation distribution: $\psi_A^M(x) = x - \frac{1-F_A(x)}{f_A(x)}$. Now, consider $x \le \widehat{v}_e$. For $x \le \widehat{v}_e$, the numerator of $\psi_A^{M\prime}(x)$ is

$$\begin{aligned}
&2[h_A^M(x)]^2 + h_A^{M\prime}(x)[H_A^M(\overline{v}) - H_A^M(x)] \\
&= 2\Big\{[1 - F_B(w_e(x))]f_A(x)\Big\}^2 - \Big\{f_B(w_e(x))w_e'(x)f_A(x) - [1 - F_B(w_e(x))]f_A'(x)\Big\}[H_A^M(\overline{v}) - H_A^M(x)] \\
&= 2\Big\{[1 - F_B(w_e(x))]f_A(x)\Big\}^2 - f_B(w_e(x))w_e'(x)f_A(x)[H_A^M(\overline{v}) - H_A^M(x)] \\
&\quad + [1 - F_B(w_e(x))]f_A'(x)[H_A^M(\overline{v}) - H_A^M(x)].
\end{aligned}$$

By the monotone hazard rate, we have $f_A'(x) \ge -\frac{f_A(x)^2}{1-F_A(x)}$, so the above can be rewritten as

$$\begin{aligned}
&2\Big\{[1 - F_B(w_e(x))]f_A(x)\Big\}^2 - f_B(w_e(x))w_e'(x)f_A(x)[H_A^M(\overline{v}) - H_A^M(x)] \\
&\quad + [1 - F_B(w_e(x))]f_A'(x)[H_A^M(\overline{v}) - H_A^M(x)] \\
&\ge 2\Big\{[1 - F_B(w_e(x))]f_A(x)\Big\}^2 - f_B(w_e(x))w_e'(x)f_A(x)[H_A^M(\overline{v}) - H_A^M(x)] \\
&\quad - [1 - F_B(w_e(x))]\frac{f_A(x)^2}{1 - F_A(x)}[H_A^M(\overline{v}) - H_A^M(x)].
\end{aligned}$$

From $w_e'(x) \le 0$, we only need to consider the first and the third terms, and by combining the two terms, we have

$$\Big\{[1 - F_B(w_e(x))]f_A(x)\Big\}^2 \left\{2 - \frac{H_A^M(\overline{v}) - H_A^M(x)}{[1 - F_A(x)][1 - F_B(w_e(x))]}\right\},$$

where

$$\begin{aligned}
H_A^M(\overline{v}) - H_A^M(x) &= \int_x^{\widehat{v}_e} \big[1 - F_B(w_e(v_i))\big] f_A(v_i)dv_i + 1 - F_A(\widehat{v}_e) \\
&= 1 - F_A(x) - \int_x^{\widehat{v}_e} F_B(w_e(v_i))f_A(v_i)dv_i.
\end{aligned}$$

The numerator of $\left\{2 - \frac{H_A^M(\overline{v}) - H_A^M(x)}{[1-F_A(x)][1-F_B(w_e(x))]}\right\}$ is

$$\begin{aligned}
&2[1-F_A(x)][1-F_B(w_e(x))] - \left[H_A^M(\overline{v}) - H_A^M(x)\right] \\
&= [1-F_A(x)][1-2F_B(w_e(x))] + \int_x^{\widehat{v}_e} F_B(w_e(v_i))f_A(v_i)dv_i \\
&= [1-F_A(x)][1-2F_B(w_e(x))] + \int_x^{\overline{v}} F_B(w_e(v_i))f_A(v_i)dv_i,
\end{aligned}$$

where the last equality follows from the fact that $F_B(w_e(v_i)) = 0$ for all $v_i > \widehat{v}_e$. Then,

$$\begin{aligned}
&[1-F_A(x)][1-2F_B(w_e(x))] + \int_x^{\overline{v}} F_B(w_e(v_i))f_A(v_i)dv_i \\
&= \int_x^{\overline{v}} \left[1-2F_B(w_e(x))\right] f_A(v_i)dv_i + \int_x^{\overline{v}} F_B(w_e(v_i))f_A(v_i)dv_i \\
&= \int_x^{\overline{v}} \left[1-2F_B(w_e(x)) + F_B(w_e(v_i))\right] f_A(v_i)dv_i.
\end{aligned}$$

Hence, if $\int_x^{\overline{v}} \left[1-2F_B(w_e(x)) + F_B(w_e(v_i))\right] f_A(v_i)dv_i > 0$, $H_A^M(x)$ is regular. ■

Proof of Theorem 1. From Proposition 1, for each $\theta_i \in M(m)$,

$$Q_A^M(v_i) = Q_A^M(\theta_i) \equiv \int_{\Theta_{-i}} q_A^M(i|\theta)d\mathcal{F}_A(v_{-i}) \times \mathcal{F}_B(w_{-i}),$$

so the expected revenue from buyer i for good A is

$$\begin{aligned}
&\int_{M(m)} \left[v_i Q_A^M(v_i) - \int_{\underline{v}}^{v_i} Q_A^M(x)dx\right] dF_A(v_i) \times F_B(w_i) \\
&= \int_{\underline{v}}^{\overline{v}} \left[v_i Q_A^M(v_i) - \int_{\underline{v}}^{v_i} Q_A^M(x)dx\right] dH_A(v_i) \\
&= \int_{\underline{v}}^{\overline{v}} Q_A^M(v_i)\psi_A^M(v_i)h_A(v_i)dv_i \\
&= \int_{\underline{v}}^{\overline{v}} Q_A^M(v_i)[1-F_B(w_e(v_i))]\psi_A^M(v_i)f_A(v_i)dv_i \\
&= \int_{\underline{v}}^{\overline{v}} \left[\int_{[\underline{v},\overline{v}]^{N-1}} \widetilde{q}_A^M(i|v)[1-F_B(w_e(v_i))]\psi_A^M(v_i)d\mathcal{F}_A(v_{-i})\right] f_A(v_i)dv_i
\end{aligned}$$

where the second equality follows from the typical step of a single dimension, and the last one from (14), $Q_A^M(v_i) = \int_{[\underline{v},\overline{v}]^{N-1}} \widetilde{q}_A^M(i|v)d\mathcal{F}_A(v_{-i})$. Now,

$$\begin{aligned}
&\sum_{i\in I} \left[\int_{\underline{v}}^{\overline{v}} Q_A^M(v_i)\psi_A^M(v_i)h_A(v_i)dv_i\right] \\
&= \sum_{i\in I} \left[\int_{[\underline{v},\overline{v}]^N} \widetilde{q}_A^M(i|v)[1-F_B(w_e(v_i))]\psi_A^M(v_i)dF_A(v_1) \times \cdots \times F_A(v_N)\right].
\end{aligned}$$

Last, the sum of allocation probabilities $\sum_{i\in I} q_A^M(i|v_i, w_e(v_i), v_{-i}, w_{-i}) \leq 1$, which leads to

$$\sum_{i\in I}\int_{[\underline{w},\overline{w}]^{N-1}} q_A^M(i|v_i, w_e(v_i), v_{-i}, w_{-i})d\mathcal{F}_B(w_{-i}) \leq 1 \Leftrightarrow \sum_{i\in I}\widetilde{q}_A^M(i|v) \leq 1 \text{ and } \widetilde{q}_A^M(i|v) \geq 0.$$

A critical difference between this and Myerson (1981) is that the result is in terms of $\widetilde{q}_A^M(i|v)$, and we have a term $[1 - F_B(w_e(v_i)]$, in addition to the modified virtual valuation function $\psi_A^M(v_i)$. If the distribution H_A^M is regular, then the virtual valuation is strictly increasing, and, furthermore, the additional term $[1 - F_B(w_e(v_i)]$ is also increasing, given that $w_e(v_i)$ is decreasing. The monotonic virtual valuation function makes Q_k^M indeed increasing for all k. The result is established. ■

Proof of Corollary 1. We show that if an **M**-mechanism satisfies IC, $u_A(v_e(w_i'), w_i') = u_A(v_e(w_i), w_i)$ for all w_i, w_i' and $u_B(w_e(v_i'), v_i') = u_B(w_e(v_i), v_i)$ for all v_i, v_i'. Consider only the former and suppose that $u_A(v_e(w_i'), w_i') \neq u_A(v_e(w_i), w_i)$ for some w_i, w_i'. If an **M**-mechanism satisfies MIC, then the separable IC implies that buyer i's payoff when he reports truthfully in (23) is greater than or equal to buyer i's payoff when he reports w_i' in (24), which yields

$$\begin{aligned}
&\int_{\underline{w}}^{w_i} Q_B^M(y)dy - \int_{\underline{w}}^{w_i'} Q_B^M(y)dy \geq w_i Q_B^M(w_i') - w_i' Q_B^M(w_i') + u_A(v_e(w_i'), w_i') - u_A(v_e(w_i), w_i) \\
&\Leftrightarrow \int_{w_i'}^{w_i} Q_B^M(y)dy \geq \int_{w_i'}^{w_i} Q_B^M(w_i')dy + u_A(v_e(w_i'), w_i') - u_A(v_e(w_i), w_i). \qquad (50)
\end{aligned}$$

We divide the proof into two cases.

Case 1. Suppose that there exists w_i such that $u_A(v_e(w_i'), w_i')$ is not continuous at w_i; that is, there exists $\epsilon > 0$ such that for a sufficiently small $\delta > 0$, $|w_i - w_i'| < \delta$, $|u_A(v_e(w_i'), w_i') - u_A(v_e(w_i), w_i)| \geq \epsilon$. Consider first $u_A(v_e(w_i'), w_i') > u_A(v_e(w_i), w_i)$ around the neighborhood. Then, from the truthful payoff (23) and the non-truthful payoff (24), if w_i' is sufficiently close to w_i, we have

$$\begin{aligned}
&\int_{w_i'}^{w_i} \left[Q_B^M(y) - Q_B^M(w_i')\right] dy < u_A(v_e(w_i'), w_i') - u_A(v_e(w_i), w_i) \\
\Leftrightarrow &\int_{w_i'}^{w_i} Q_B^M(y)dy < \int_{w_i'}^{w_i} Q_B^M(w_i')dy + u_A(v_e(w_i'), w_i') - u_A(v_e(w_i), w_i),
\end{aligned}$$

which violates MIC in (50). If, on the other hand, $u_A(v_e(w_i'), w_i') < u_A(v_e(w_i), w_i)$ around the neighborhood, we can consider the case that w_i' is a true valuation to show the violation of MIC.

Case 2. $u_A(v_e(w_i'), w_i')$ is continuous. Now suppose that

$$\lim_{w_i' \to w_i+} \frac{u_A(v_e(w_i'), w_i') - u_A(v_e(w_i), w_i)}{w_i - w_i} \neq 0 \text{ or } \lim_{w_i' \to w_i-} \frac{u_A(v_e(w_i'), w_i') - u_A(v_e(w_i), w_i)}{w_i - w_i} \neq 0.$$

Then, the difference between the non-truthful payoff in (24) and the truthful payoff (23) is given as

$$\int_{\underline{v}}^{v_i} Q_A^M(x)dx + w_i Q_B^M(w_i') - \left[w_i' Q_B^M(w_i') - \int_{\underline{w}}^{w_i'} Q_B^M(y)dy\right] + u_A(v_e(w_i'), w_i') + u_B(v_i, w_e(v_i)) - e$$
$$- u(v_i, w_i)$$
$$= w_i Q_B^M(w_i') - w_i' Q_B^M(w_i') - \int_{\underline{w}}^{w_i} Q_B^M(y)dy + \int_{\underline{w}}^{w_i'} Q_B^M(y)dy + u_A(v_e(w_i'), w_i') - u_A(v_e(w_i), w_i)$$
$$= \int_{w_i'}^{w_i} Q_B^M(w_i')dy - \int_{w_i'}^{w_i} Q_B^M(y)dy + u_A(v_e(w_i'), w_i') - u_A(v_e(w_i), w_i).$$

Now, divide the above payoff difference by $w_i' - w_i$, which yields

$$\frac{\int_{w_i'}^{w_i} Q_B^M(w_i')dy - \int_{w_i'}^{w_i} Q_B^M(y)dy}{w_i' - w_i} + \frac{u_A(v_e(w_i'), w_i') - u_A(v_e(w_i), w_i)}{w_i' - w_i}.$$

Then, without loss of generality, consider $\lim_{w_i' \to w_i+} \frac{u_A(v_e(w_i'), w_i') - u_A(v_e(w_i), w_i)}{w_i - w_i} \neq 0$. By $Q_B^M(y) = G_B(y)$ from Theorem 1 and Leibniz rule,

$$\lim_{w_i' \to w_i+} \frac{\int_{w_i'}^{w_i} Q_B^M(w_i')dy - \int_{w_i'}^{w_i} Q_B^M(y)dy}{w_i' - w_i} = 0,$$

which implies that there exists a sufficiently small neighborhood around w_i such that a misreport w_i' that belongs to the neighborhood increases buyer i's payoff, so we have the violation of MIC in (50). The proof is complete, and the result follows from (17). ■

Proof of Lemma 3. Consider the numerator of (38). By incorporating $G_A(x) = F_A(x)^{N-1}$ and $g_A(x) = (N-1)F_A(x)^{N-2}f_A(x)$, it can be rewritten as

$$r_A F_A(r_A) G_A(r_A) - \int_{\underline{v}}^{r_A} v_i g_A(v_i) F_A(v_i) dv_i$$
$$= r_A F_A(r_A)^N - r_A \frac{N-1}{N} F_A(r_A)^N + \int_{\underline{v}}^{r_A} \frac{N-1}{N} F_A(v_i)^N dv_i$$
$$= \frac{r_A F_A(r_A)^N}{N} + \frac{N-1}{N} \int_{\underline{v}}^{r_A} F_A(v_i)^N dv_i,$$

where, by the integration by parts, $\int_{\underline{v}}^{r_A} v_i g_A(v_i) F_A(v_i) dv_i = r_A \frac{N-1}{N} F_A(r_A)^N - \int_{\underline{v}}^{r_A} \frac{N-1}{N} F_A(v_i)^N dv_i$.

Hence,

$$
\begin{aligned}
&\frac{r_A F_A(r_A) G_A(r_A) - \int_{\underline{v}}^{r_A} v_i g_A(v_i) F_A(v_i) dv_i}{\int_{\underline{v}}^{r_A} G_A(v_i) dv_i} \\
&= \frac{r_A F_A(r_A)^N + (N-1) \int_{\underline{v}}^{r_A} F_A(v_i)^N dv_i}{N \int_{\underline{v}}^{r_A} F_A(v_i)^{N-1} dv_i} \\
&= \frac{r_A F_A(r_A)^N + N \int_{\underline{v}}^{r_A} F_A(v_i)^N dv_i}{N \int_{\underline{v}}^{r_A} F_A(v_i)^{N-1} dv_i} - \frac{\int_{\underline{v}}^{r_A} F_A(v_i)^N dv_i}{N \int_{\underline{v}}^{r_A} F_A(v_i)^{N-1} dv_i} \\
&= \frac{r_A \int_{\underline{v}}^{r_A} [N F_A(v_i)^{N-1} f_A(v_i)] dv_i + N \int_{\underline{v}}^{r_A} F_A(v_i)^N dv_i}{N \int_{\underline{v}}^{r_A} F_A(v_i)^{N-1} dv_i} - \frac{\int_{\underline{v}}^{r_A} F_A(v_i)^N dv_i}{N \int_{\underline{v}}^{r_A} F_A(v_i)^{N-1} dv_i} \\
&= \frac{\int_{\underline{v}}^{r_A} [r_A F_A(v_i)^{N-1} f_A(v_i)] dv_i + \int_{\underline{v}}^{r_A} F_A(v_i)^N dv_i}{\int_{\underline{v}}^{r_A} F_A(v_i)^{N-1} dv_i} - \frac{\int_{\underline{v}}^{r_A} F_A(v_i)^N dv_i}{N \int_{\underline{v}}^{r_A} F_A(v_i)^{N-1} dv_i}.
\end{aligned}
\tag{51}
$$

First, the second term of the formula above satisfies that for all $n \geq 2$,

$$
F_A(v_i)^N < F_A(v_i)^{N-1} < \frac{N}{N-1} F_A(v_i)^{N-1}
$$

so for all $N \geq 2$,

$$
\frac{\int_{\underline{v}}^{r_A} F_A(v_i)^N dv_i}{N \int_{\underline{v}}^{r_A} F_A(v_i)^{N-1} dv_i} < \frac{1}{N-1} \Leftrightarrow -\frac{\int_{\underline{v}}^{r_A} F_A(v_i)^N dv_i}{N \int_{\underline{v}}^{r_A} F_A(v_i)^{N-1} dv_i} > -\frac{1}{N-1}. \tag{52}
$$

Now, the first term can be rewritten as

$$
\frac{\int_{\underline{v}}^{r_A} [r_A F_A(v_i)^{N-1} f_A(v_i)] dv_i + \int_{\underline{v}}^{r_A} F_A(v_i)^N dv_i}{\int_{\underline{v}}^{r_A} F_A(v_i)^{N-1} dv_i} = \frac{\int_{\underline{v}}^{r_A} F_A(v_i)^{N-1} [r_A f_A(v_i) + F_A(v_i)] dv_i}{\int_{\underline{v}}^{r_A} F_A(v_i)^{N-1} dv_i},
$$

where

$$
\begin{aligned}
\frac{\int_{\underline{v}}^{r_A} F_A(v_i)^{N-1} [r_A f_A(v_i) + F_A(v_i)] dv_i}{\int_{\underline{v}}^{r_A} F_A(v_i)^{N-1} dv_i} &> \frac{\int_{\underline{v}}^{r_A} F_A(v_i)^{N-1} [v_i f_A(v_i) + F_A(v_i)] dv_i}{\int_{\underline{v}}^{r_A} F_A(v_i)^{N-1} dv_i} \\
&= \frac{\int_{\underline{v}}^{r_A} F_A(v_i)^{N-1} [v_i f_A(v_i) - 1 + F_A(v_i) + 1] dv_i}{\int_{\underline{v}}^{r_A} F_A(v_i)^{N-1} dv_i} \\
&= \frac{\int_{\underline{v}}^{r_A} F_A(v_i)^{N-1} f_A(v_i) \left[v_i - \frac{1 - F_A(v_i)}{f_A(v_i)} \right] dv_i}{\int_{\underline{v}}^{r_A} F_A(v_i)^{N-1} dv_i} + 1.
\end{aligned}
$$

Note that, by the integration by parts,

$$
\begin{aligned}
&\int_{\underline{v}}^{r_A} F_A(v_i)^{N-1} f_A(v_i) \left[v_i - \frac{1 - F_A(v_i)}{f_A(v_i)} \right] dv_i \\
&= \frac{1}{N} F_A(v_i)^N \left[v_i - \frac{1 - F_A(v_i)}{f_A(v_i)} \right] \Bigg|_{\underline{v}}^{r_A} - \int_{\underline{v}}^{r_A} \frac{1}{N} F_A(v_i)^N \left[v_i - \frac{1 - F_A(v_i)}{f_A(v_i)} \right]' dv_i \\
&= -\int_{\underline{v}}^{r_A} \frac{1}{N} F_A(v_i)^N \left[v_i - \frac{1 - F_A(v_i)}{f_A(v_i)} \right]' dv_i.
\end{aligned}
$$

Hence, the first term satisfies the following inequality such that

$$\frac{\int_{\underline{v}}^{r_A} F_A(v_i)^{N-1}[r_A f_A(v_i) + F_A(v_i)]dv_i}{\int_{\underline{v}}^{r_A} F_A(v_i)^{N-1}dv_i} > -\frac{\int_{\underline{v}}^{r_A} F_A(v_i)^N \left[v_i - \frac{1-F_A(v_i)}{f_A(v_i)}\right]' dv_i}{N\int_{\underline{v}}^{r_A} F_A(v_i)^{N-1}dv_i} + 1.$$

For each $N \geq 2$,

$$\left[v_i - \frac{1-F_A(v_i)}{f_A(v_i)}\right]' F_A(v_i)^N < \frac{N}{N-1}\left[v_i - \frac{1-F_A(v_i)}{f_A(v_i)}\right]' F_A(v_i)^N \leq \frac{N}{N-1}\overline{B}F_A(r_A)F_A(v_i)^{N-1}$$

where for each $v_i \in [\underline{v}, r_A]$, $\left[v_i - \frac{1-F_A(v_i)}{f_A(v_i)}\right]' \leq \overline{B}$ for some $\overline{B}$, which yields

$$\frac{\int_{\underline{v}}^{r_A} F_A(v_i)^N \left[v_i - \frac{1-F_A(v_i)}{f_A(v_i)}\right]' dv_i}{N\int_{\underline{v}}^{r_A} F_A(v_i)^{N-1}dv_i} < \frac{\overline{B}F_A(r_A)}{N-1} \Leftrightarrow -\frac{\int_{\underline{v}}^{r_A} F_A(v_i)^N \left[v_i - \frac{1-F_A(v_i)}{f_A(v_i)}\right]' dv_i}{N\int_{\underline{v}}^{r_A} F_A(v_i)^{N-1}dv_i} > -\frac{\overline{B}F_A(r_A)}{N-1}.$$

Thus, for all $N \geq 2$, the inequality of the first term becomes

$$\frac{\int_{\underline{v}}^{r_A} F_A(v_i)^{N-1}[r_A f_A(v_i) + F_A(v_i)]dv_i}{\int_{\underline{v}}^{r_A} F_A(v_i)^{N-1}dv_i} > 1 - \frac{\overline{B}F_A(r_A)}{N-1}. \tag{53}$$

By combining the first term (53) and the second term (52), for all $N \geq 2$, the ratio (51) satisfies the inequality

$$\frac{r_A F_A(r_A)G_A(r_A) - \int_{\underline{v}}^{r_A} v_i g_A(v_i)F_A(v_i)dv_i}{\int_{\underline{v}}^{r_A} G_A(v_i)dv_i} > 1 - \frac{\overline{B}F_A(r_A)}{N-1} - \frac{1}{N-1}.$$

In the limit, we have

$$\lim_{N\to\infty} \frac{r_A F_A(r_A)G_A(r_A) - \int_{\underline{v}}^{r_A} v_i g_A(v_i)F_A(v_i)dv_i}{\int_{\underline{v}}^{r_A} G_A(v_i)dv_i} \geq 1 - \lim_{N\to\infty}\frac{\overline{B}F_A(r_A)}{N-1} - \lim_{N\to\infty}\frac{1}{N-1} = 1.$$

This completes the proof. ■

Proof of Theorem 2. Similarly, for good B, we have

$$\int_{\underline{w}}^{r_B} G_B(y)dy\left[\left(1 - F_A(r_A)\right)D_B(N) + F_A(r_A)F_B(r_B) - 1\right],$$

where

$$D_B(N) \equiv \frac{[r_B G_B(r_B)F_B(r_B) - \int_{\underline{w}}^{r_B} w_i g_B(w_i)F_B(w_i)dw_i}{\int_{\underline{w}}^{r_B} G_B(y)dy}.$$

Now, we have at least the following membership fee $[1 - F_A(r_A)F_B(r_B)]\widehat{e}_N$, where recall $\widehat{e}_N \equiv \int_{\underline{w}}^{r_B} G_B(y)dy$ and note that over the sequence, this value $e = \widehat{e}_N$ can change as N changes. Together, we have

$$\begin{aligned}
&\int_{\underline{v}}^{r_A} G_A(x)dx \left[\big(1 - F_B(r_B)\big)D_A(N) + F_A(r_A)F_B(r_B) - 1\right] \\
&+ \int_{\underline{w}}^{r_B} G_B(y)dy \left[\big(1 - F_A(r_A)\big)D_B(N) + F_A(r_A)F_B(r_B) - 1\right] \\
&+ [1 - F_A(r_A)F_B(r_B)]\widehat{e}_N \\
= \widehat{e}_N &\left[\big(1 - F_B(r_B)\big)D_A(N) + \big(1 - F_A(r_A)\big)D_B(N) + F_A(r_A)F_B(r_B) - 1\right].
\end{aligned}$$

With the above formula, we examine the limit of the part inside of the term:

$$\lim_{N\to\infty}[\big(1-F_B(r_B)\big)D_A(N)+\big(1-F_A(r_A)\big)D_B(N)+F_A(r_A)F_B(r_B)-1] \geq \big(1-F_A(r_A)\big)\big(1-F_B(r_B)\big) > 0,$$

which establishes the theorem. ■

Proof of Theorem 3. We first show that as $N \to \infty$, $|G_A(x) - G_B(x)| \to 0$ for all x. By the mean value theorem, there exists $k < 1$ such that

$$|G_A(x) - G_B(x)| = |F_A(x)^{N-1} - F_B(x)^{N-1}| = |F_A(x) - F_B(x)|(N-1)k^{N-2}.$$

Furthermore,

$$\begin{aligned}
&\left|\int_{\underline{v}}^{r_A} G_A(x)dx - \int_{\underline{w}}^{r_B} G_B(y)dy\right| \\
\leq &\left|\int_{\underline{v}}^{r_A} G_A(x)dx - \int_{\underline{v}}^{r_B} G_A(x)dx\right| + \left|\int_{\underline{v}}^{r_B} G_A(x)dx - \int_{\underline{v}}^{r_B} G_B(x)dx\right| \\
= &|r_A - r_B|G_A(r_C) + \left|\int_{\underline{v}}^{r_B} G_A(x)dx - \int_{\underline{v}}^{r_B} G_B(x)dx\right|,
\end{aligned}$$

where the last equality follows from the mean value theorem given $r_C \in [\min\{r_A, r_B\}, \max\{r_A, r_B\}]$. If the condition (41) is satisfied, then (40) is

$$\begin{aligned}
&\int_{\underline{v}}^{r_A} G_A(x)dx \left[\big(1 - F_B(r_B)\big)D_A(N) + F_A(r_A)F_B(r_B) - 1\right] \\
&+ \widehat{e}_N \left[\big(1 - F_A(r_A)\big)D_B(N) + F_A(r_A)F_B(r_B) - 1\right] + [1 - F_A(\widehat{v}_e)F_B(r_B)]\widehat{e}_N \\
\geq &\int_{\underline{v}}^{r_A} G_A(x)dx\big(1 - F_B(r_B)\big)D_A(N) + \widehat{e}_N \left[\big(1 - F_A(r_A)\big)D_B(N) + F_A(r_A)F_B(r_B) - 1\right] \\
\geq &\widehat{e}_N\big(1 - F_B(r_B)\big)D_A(N) + \widehat{e}_N \left[\big(1 - F_A(r_A)\big)D_B(N) + F_A(r_A)F_B(r_B) - 1\right],
\end{aligned}$$

where the last inequality follows from $\int_{\underline{v}}^{r_A} G_A(x)dx > \int_{\underline{w}}^{r_B} G_B(y)dy = \widehat{e}_N$. ■

Proof of Theorem 4. Fix N, and consider only good A. By the integration by parts,

$$\begin{aligned}\int_{\underline{v}}^{r_A}\left[\int_{\underline{v}}^{v_i} x dG_A(x)\right] dF_A(v_i) &= \left[\int_{\underline{v}}^{v_i} x dG_A(x)\right] F_A(v_i)\Big|_{\underline{v}}^{r_A} - \int_{\underline{v}}^{r_A} v_i g_A(v_i) F_A(v_i) dv_i \\ &= \left[\int_{\underline{v}}^{r_A} x dG_A(x)\right] F_A(r_A) - \int_{\underline{v}}^{r_A} v_i g_A(v_i) F_A(v_i) dv_i \\ &= F_A(r_A)\left[r_A G_A(r_A) - \int_{\underline{v}}^{r_A} G_A(x) dx\right] - \int_{\underline{v}}^{r_A} v_i g_A(v_i) F_A(v_i) dv_i.\end{aligned}$$

Hence, by incorporating the degree of first-order stochastic dominance with $n \in \mathbb{N}$ such as $F_A(v_i)^n$, (43) satisfies the following inequality:

$$\begin{aligned}&\int_{\underline{v}}^{r'_A}\left[\int_{\underline{v}}^{v_i} x dG_A(x)\right] dF_A(v_i)^n - (1 - F_A(r'_A)^n)\int_{\underline{v}}^{r'_A} G_A(x) dx \\ &\leq r'_A F_A(r'_A)^n G_A(r'_A) - (1 - F_A(r'_A)^n)\int_{\underline{v}}^{r'_A} G_A(x) dx,\end{aligned}$$

which yields (44). A similar procedure can be applied for good B. In addition, for each n, now denote a membership $e = \widehat{e}_N(n)$ such that

$$\widehat{e}_N(n) \equiv \int_{\underline{v}}^{r'_A} G_A(x) dx.$$

Then, the total difference yields

$$\begin{aligned}&\int_{\underline{v}}^{r'_A}\left[\int_{\underline{v}}^{v_i} x dG_A(x)\right] dF_A(v_i)^n - (1 - F_A(r'_A)^n)\int_{\underline{v}}^{r'_A} G_A(x) dx \\ &\quad + \int_{\underline{w}}^{r'_B}\left[\int_{\underline{w}}^{w_i} x dG_B(x)\right] dF_B(w_i)^n - (1 - F_B(r'_B)^n)\int_{\underline{w}}^{r'_B} G_B(y) dy + \widehat{e}_N(n) H_A^M(\overline{v}) \\ &\leq r'_A F_A(r'_A)^n G_A(r_A) - (1 - F_A(r'_A)^n)\int_{\underline{v}}^{r'_A} G_A(x) dx \\ &\quad + r'_B F_B(r'_B)^n G_B(r_B) - (1 - F_B(r'_B)^n)\int_{\underline{w}}^{r'_B} G_B(y) dy + \widehat{e}_N(n) \\ &= r'_A F_A(r'_A)^{nN} - (1 - F_A(r'_A)^n)\int_{\underline{v}}^{r'_A} F_A(x)^{n(N-1)} dx \\ &\quad + r'_B F_B(r'_B)^{nN} - (1 - F_B(r'_B)^n)\int_{\underline{w}}^{r'_B} F_B(y)^{n(N-1)} dy + \widehat{e}_N(n).\end{aligned}$$

The above equation can be rewritten as

$$\widehat{e}_N(n)\left[\frac{r'_A F_A(r'_A)^{nN}}{\int_{\underline{v}}^{r'_A} F_A(x)^{n(N-1)} dx} + \frac{r'_B F_B(r'_B)^{nN}}{\int_{\underline{w}}^{r'_B} F_B(y)^{n(N-1)} dy} + F_A(r'_A)^n + F_B(r'_B)^n - 1\right]. \tag{54}$$

Consider the first term in the bracket and rewrite it such that

$$\frac{r'_A F_A(r'_A)^{nN}}{\int_{\underline{v}}^{r'_A} F_A(x)^{n(N-1)}dx} = \frac{r'_A F_A(r'_A)^{n(N-1)}}{\int_{\underline{v}}^{r'_A} F_A(x)^{n(N-1)}dx} F_A(r'_A)^n = \frac{r'_A F_A(r'_A)^{n(N-1)}}{n\int_{\underline{v}}^{r'_A} F_A(x)^{n(N-1)}dx} nF_A(r'_A)^n. \quad (55)$$

Note that $nF_A(r'_A)^n$ converges to zero, but we are not sure whether or not $\frac{r'_A F_A(r'_A)^{n(N-1)}}{n\int_{\underline{v}}^{r'_A} F_A(x)^{n(N-1)}dx}$ will converge to infinity faster than the rate in which $nF_A(r'_A)^n$ converges to zero. To tackle it, consider

$$\int_{\underline{v}}^{r'_A} x\big[n(N-1)F_A(x)^{n(N-1)-1}f_A(x)\big]dx,$$

which can be rewritten as, using the integration by parts,

$$\int_{\underline{v}}^{r'_A} x\big[n(N-1)F_A(x)^{n(N-1)-1}f_A(x)\big]dx = r'_A F_A(r'_A)^{n(N-1)} - \int_{\underline{v}}^{r'_A} F_A(x)^{n(N-1)}dx$$

$$\Leftrightarrow r'_A F_A(r'_A)^{n(N-1)} = \int_{\underline{v}}^{r'_A} x\big[n(N-1)F_A(x)^{n(N-1)-1}f_A(x)\big]dx + \int_{\underline{v}}^{r'_A} F_A(x)^{n(N-1)}dx.$$

With the above, (55) can be reformulated as

$$\begin{aligned}
&\frac{r'_A F_A(r'_A)^{n(N-1)}}{n\int_{\underline{v}}^{r'_A} F_A(x)^{n(N-1)}dx} nF_A(r'_A)^n \\
&= \frac{\int_{\underline{v}}^{r'_A} x\big[n(N-1)F_A(x)^{n(N-1)-1}f_A(x)\big]dx + \int_{\underline{v}}^{r'_A} F_A(x)^{n(N-1)}dx}{n\int_{\underline{v}}^{r'_A} F_A(x)^{n(N-1)}dx} nF_A(r'_A)^n \\
&= \left\{(N-1)\frac{\int_{\underline{v}}^{r'_A} x\big[F_A(x)^{n(N-1)-1}f_A(x)\big]dx}{\int_{\underline{v}}^{r'_A} F_A(x)^{n(N-1)}dx} + \frac{1}{n}\right\} nF_A(r'_A)^n,
\end{aligned}$$

where we find that there exists an upper bound $\overline{B}_A > 0$ such that for all $x \in [\underline{v}, \overline{v}]$, $x\frac{f_A(x)}{F_A(x)} \leq \overline{B}_A$, which implies

$$\begin{aligned}
&x\frac{f_A(x)}{F_A(x)}F_A(x)^{n(N-1)} \leq \overline{B}_A F_A(x)^{n(N-1)} \text{ for all } x \in [\underline{v}, \overline{v}] \\
\Rightarrow &\int_{\underline{v}}^{r'_A} x\big[F_A(x)^{n(N-1)-1}f_A(x)\big]dx \leq \overline{B}_A \int_{\underline{v}}^{r'_A} F_A(x)^{n(N-1)}dx.
\end{aligned}$$

Then, with the bound, the first term (55) converges to zero from $\lim_{n\to\infty} nF_A(r'_A)^n = 0$. The same procedure applies to the second term for good B in the bracket. Together, for a sufficiently large n, (54) is negative, which establishes the result. ■

Proof of Corollary 3. With ex-ante reserve prices or absolute exclusion, the following

inequality, equivalent to (43), holds:

$$\int_{\delta_A}^{r'_A}\int_{w_{(e,\delta)}(x)}^{\overline{w}}\left[\delta_A G_A(\delta_A)+\int_{\delta_A}^{v_i} x dG_A(x)\right]dF_B(w_i)\times F_A(v_i)-(1-F_A(r_A))\int_{\delta_A}^{r_A}G_A(x)dx$$
$$\leq \int_{\delta_A}^{r'_A}\left[\delta_A G_A(\delta_A)+\int_{\delta_A}^{v_i} x dG_A(x)\right]dF_A(v_i)-(1-F_A(r_A))\int_{\delta_A}^{r_A}G_A(x)dx.$$

We examine the first term above. The first component of it can be rewritten as

$$\int_{\delta_A}^{r'_A}\left[\delta_A G_A(\delta_A)\right]dF_A(v_i)=(F_A(r'_A)-F_A(\delta_A))\delta_A G_A(\delta_A),$$

and the second component of it can be rewritten as

$$\int_{\delta_A}^{r'_A}\left[\int_{\delta_A}^{v_i} x dG_A(x)\right]dF_A(v_i)=\left[r'_A G_A(r'_A)-\delta_A G_A(\delta_A)-\int_{\delta_A}^{r'_A}G_A(x)dx\right]F_A(r'_A)$$
$$-\int_{\delta_A}^{r_A} v_i g_A(v_i)F_A(v_i)dv_i.$$

Together, by cancelling out $F_A(r'_A)\delta_A G_A(\delta_A)$, we have

$$r'_A G_A(r'_A)F_A(r'_A)-F_A(\delta_A)\delta_A G_A(\delta_A)-\int_{\delta_A}^{r'_A}G_A(x)dxF_A(r'_A)-\int_{\delta_A}^{r_A} v_i g_A(v_i)F_A(v_i)dv_i$$
$$-(1-F_A(r'_A))\int_{\delta_A}^{r'_A}G_A(x)dx\leq r'_A G_A(r'_A)F_A(r'_A)-(1-F_A(r'_A))\int_{\delta_A}^{r'_A}G_A(x)dx.$$

Then, by incorporating the degree of first-order stochastic dominance with $n\in\mathbb{N}$ such as $F_A(v_i)^n$, the inequality (45) is satisfied. The same procedure applies to good B. The remaining proof is identical to Theorem 4. ■

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