Membership Mechanism

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Abstract

This paper studies an environment in which a seller seeks to sell two different items to buyers. The seller designs a membership mechanism that assigns positive allocations to members only. Exploiting the restrictive set, the seller finds a revenue-maximizing incentive compatible mechanism. We first establish the optimal allocation rule for this membership mechanism given a regularity condition for a modified valuation distribution reflecting the set, which provides the existence of a member set and the optimal payment rule. The optimal allocation enables us to compare the membership with separate selling of the two items, suggesting conditions under which the membership dominates the separate selling: interplay between the number of bidders and the degree of the stochastic dominance of valuation distributions.

Keywords and Phrases: Mechanism design, Multidimensional screening, Auction

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1 Introduction

Memberships are ubiquitous. One common usage of the group formations, especially with the recent rise of electronic commerce, is sales memberships. A sales membership allows its members to purchase multiple items in return for membership fees collected from them. Despite the popularity of this daily business practice, the rationale behind the membership, in the area of selling multiple items, has not been investigated thoroughly. That is, why should sellers prefer a membership to “separate selling” (Myerson (1981))?\(^1\)

The single-item optimal sales mechanism has been extended to consider multidimensional screening not only to respond to its purely theoretical challenges but to encompass real-world sales environments. The setting with buyers having multidimensional valuations for multiple items being natural, its analytical difficulty and anomaly are well documented, even for mild extensions, such as only two items: there is no full characterization for the optimal allocation.\(^2\)

This paper suggests a new direction for multidimensional screening. First, in the model, each buyer participates in the purchase of two items separately, to reflect the practice in reality.\(^3\) We assume the buyer’s separate participation, after becoming a member, not just to simplify the analysis; it is simply not our main focus in this paper.\(^4\) The driving force behind the membership mechanism is that the seller chooses a restrictive member set to exploit the buyer’s multidimensional type, compared with the separate selling. But it must resolve an existence problem of a member set before analyzing the comparison. A membership mechanism assigns positive allocations to members only. As a result, the mechanism’s allocation and payment may depend on a member set that the seller adopts, which in turn

\(^1\)The optimal mechanism of Myerson (1981) with a single item can be implemented through an auction, but, in reality, posted prices are still commonly used, due to other institutional considerations. See, e.g., Wang (1993). Likewise, this membership mechanism is not only directly applicable to a type of membership with auctions such as eBay but also has relevant insights for other types of sales membership.

\(^2\)For instance, the bundling’s dominance by McAfee, McMillan and Whinston (1989) is established for “mixed” bundling, posting both individual prices and bundle prices. The dominance between “pure” bundling and the separate selling is, however, not determinant, as observed by Thanassoulis (2004) and Manelli and Vincent (2006).

\(^3\)This requires direct mechanisms to represent such indirect mechanisms.

\(^4\)We use the term “separate selling” by the seller differently from the term “separate participation” by the buyers in this paper. The former refers to applying the optimal mechanism of Myerson (1981) to each item, but the latter implies separable incentive compatibility for each buyer in a direct mechanism.
generates each buyer's payoff, determining his or her willingness to become a member or not, or a member set. Hence, in nature, the existence of a member set satisfying such incentive compatibility involves a fixed point type argument.

The first main result shows that the highest valuation for an item among members wins the item if a modified virtual valuation distribution reflecting the restrictive member set satisfies monotonicity. The finding resembles a single dimension, but we must find the optimal allocation without being able to pin down the payoff of a buyer with the lowest valuation in one item; in a multidimension, the buyer’s payoff and his decision to be a member depend on the other good’s valuation as well, unlike in a single dimension. The optimal allocation solves the existence problem, implying a “normalization” of the optimal payment rule. Essentially, the problem of choosing a member set and a membership fee is reduced to choosing a membership fee, considering the incentive compatibility condition.

The membership’s dominance over the separate selling requires a systematic approach. A direct approach that compares closed-form solutions, shown for an illustrative example with an uniform distribution, is futile with general environments, such as non-uniform distributions (Section 5). We find a link between the two mechanisms, membership and separate selling, by connecting a member set’s intercept to an optimal reserve price from the separate selling. In a membership, unlike the separate selling, a bidder whose valuation is low in one dimension can be a member if the other valuation is sufficiently high. Hence, those bidder types’ expected payment becomes an additional source of the seller’s revenue. On the other hand, with the membership mechanism, without a “reserve” price, lower winning bids can realize, which decreases the seller’s revenue. In other words, a membership induces a larger set of types to participate, but some of them can win with lower bids. As the number of bidders increases, the former effect dominates the latter; the membership generates a higher revenue than the separate selling.

The opposite case is also examined. The separate selling dominates the optimal membership mechanism if bidders’ valuation distributions become sufficiently stochastic dominant. Under the condition, in a membership, the revenue decrease from low bids dominates the additional revenue from low valuations when valuation distributions include more high valuations.

The well-known equivalence between a reserve price and an (interim) entry fee in a single dimension no longer holds in multidimensional types: a single entry fee, which we call a membership fee in this model, cannot capture two different reserve prices in two dimensions.
This failure makes room for the comparison between the membership with a single fee and the separate selling with two separate reserve prices. This new approach can shed light on how a mechanism in multidimensional types can outperform the mechanism by Myerson (1981).

Although the robustness of a non-negligible set of not-participating buyer types in multidimensions was well established by Armstrong (1996) and Rochet and Choné (1998), the full characterization of the optimal allocation for participating buyer types is generally intractable. The study on this restricted domain enables us to find the optimal allocation for members of this model so that we can compare its performance with the separate selling.

The anomaly of multidimensional screening was reported by early pioneers, Thanassoulis (2004) and Manelli and Vincent (2006), and further important observations were made by Hart and Reny (2015) and Hart and Nisan (2017). The complexities are not our main concern in this model with the buyer’s separate participation. Yet, it is critical for the seller of this paper to utilize a modified joint distribution that incorporates a member set, despite the independence of two valuations, which is a sharp contrast with Carroll (2017), in which the principal has beliefs only about each marginal distribution. The focus of this paper, the seller exploiting a multidimension with a restrictive set, makes our problem differ from the literature on bundling (see Manelli and Vincent (2006), Manelli and Vincent (2007) and Hart and Nisan (2017)).

This paper is also related to auctions with budget constraints, as a membership fee constrains a buyer’s feasible choices. We study an incomplete information model with two items, unlike the complete information case by Che and Gale (1998), Che and Gale (2000) and Benoit and Krishna (2001), and the single-item case by Pai and Vohra (2014). The setting with incomplete information and two items makes sense of the comparison between its performance and that of its counterpart, the separate selling by Myerson (1981). In addition, choosing a membership fee and the constraint it imposes is the seller’s endogenous variable, not an exogenous environment like a budget constraint.

The model is in Section 2 and the membership mechanism is introduced in Section 3. Sections 4 and 5 provide the main results: the optimal characterization of the membership and its dominance over the separate selling. The dominance of the separate selling is found Section 6 and the concluding remarks are in the last section. All the proofs are collected in an appendix.
2 Model

One seller seeks to sell two non-identical items to \( N \geq 2 \) potential buyers. Each of the two items, good \( A \) and good \( B \), is a single unit of an indivisible good. Buyer \( i \in I \equiv \{1, ..., N\} \) receives value \( v_i \) from good \( A \), and value \( w_i \) from good \( B \). We call buyer \( i \)'s valuations \((v_i, w_i)\) buyer \( i \)'s type, denoted by \( \theta_i \equiv (v_i, w_i) \in \Theta_i \equiv [v, \overline{v}] \times [w, \overline{w}] \). The two goods' valuations are not related; \( v_i \) and \( w_i \) are independently drawn from \([v, \overline{v}]\), \( v > v \geq 0 \), and \([w, \overline{w}]\), \( \overline{w} > w \geq 0 \), respectively. Each good’s valuations, across buyers, are independently and identically drawn, according to a differentiable cumulative distribution function \( F_k \) with its density \( f_k > 0 \), for \( k \in \{A, B\} \).

An outcome \( x = (x_A, x_B) \) specifies which good is assigned to a buyer, or not sold, with a set of outcomes \( X \equiv \{0\} \cup I \times \{0\} \cup I \).\(^5\) The probability that outcome \( x \) occurs, denoted by \( q(x) \), generates its marginal probabilities: for \( x_A, x_B \in I \), \( q_A(x_A) \equiv \sum_{x_B \in \{0\} \cup I} q(x) \) the probability of good \( A \) sold to buyer \( x_A \), and \( q_B(x_B) \equiv \sum_{x_A \in \{0\} \cup I} q(x) \) the probability of good \( B \) sold to buyer \( x_B \). Each buyer’s total benefit from trade being the sum of the two valuations, buyer \( i \) obtains expected payoff \( q_A(i)v_i + q_B(i)w_i - t_i \) if he purchases good \( A \) with probability \( q_A(i) \) and good \( B \) with probability \( q_B(i) \), by paying a transfer \( t_i \) to the seller, and the seller obtains revenue \( \sum_{i \in I} t_i \) if for each \( i \in I \), buyer \( i \) transfers \( t_i \) to him.

For each \( k \in \{A, B\} \), we assume that \( F_k \) satisfies the standard monotone hazard rate condition: \( \frac{1-F_k(x)}{f_k(x)} \) is non-increasing, and that the type distribution \( F_k \) is common knowledge among players. Finally, each buyer’s reservation payoff is normalized as zero.

3 Membership mechanism

A direct mechanism can be defined with measurable functions \((q, t_1, ..., t_N)\), where \( q : \Theta \rightarrow \Delta(X) \) and \( t_i : \Theta \rightarrow \mathbb{R} \), with a set of all probability distributions over \( X \), denoted by \( \Delta(X) \), and a type profile is \( \theta \equiv (\theta_1, ..., \theta_N) \in \Theta \equiv \Theta_1 \times \cdots \times \Theta_N \). If buyer \( i \) reports \( \theta_i \) for all \( i \in I \), the seller commits to an outcome \( x \) with probability \( q(x|\theta) \) by collecting a transfer \( t_i(\theta) \) from buyer \( i \).

We restrict it to define a membership mechanism, called a direct \( M \)-mechanism, which allows only a member to purchase two goods separately. Buyer \( i \) becomes a member if his reported type satisfies a criterion \( m(\theta_i) \geq 0 \), by paying a membership fee \( e \in \mathbb{R} \). The

\(^5\)For example, \( x = (0, 2) \) indicates that good \( A \) remains with the seller and good \( B \) is sold to buyer 2.
Figure 1: Membership of example 1

function $m$ is continuous, and it is, weakly, monotonic, which includes Example 1, $m(\theta_i) = \max\{v_i, w_i\} - \frac{1}{2}$ in figure 1. A set of types satisfying the criterion, a set of member types, is denoted by $M(m)$ such that

$$\quad (1) \quad M(m) \equiv \{\theta_i \in \Theta_i : m(\theta_i) \geq 0\}.$$

The set is closed and connected from the assumptions on $m$. In addition, denote by $\overline{M}(m) \equiv \{\theta_i \in \Theta_i : m(\theta_i) = 0\}$ a set of member types that are qualified just enough to satisfy the criterion.

With a set of member types $M(m)$, or simply $M$, an $M$-mechanism has an allocation rule $q^M(x|\theta)$ and its marginal probabilities $q^M_A(x_A|\theta) \equiv \sum_{x_B \in \{0\} \cup I} q^M(x|\theta)$ and $q^M_B(x_B|\theta) \equiv \sum_{x_A \in \{0\} \cup I} q^M(x|\theta)$. The separate purchases of the two goods for the mechanism require two different transfers: $t^M = (t^M_A, t^M_B)$ such that for each $k \in \{A, B\}$, $t^M_k : \Theta \rightarrow \mathbb{R}^N$, where

Formally, $(v'_i, w'_i) \gg (v_i, w_i)$ implies $m(v'_i, w'_i) > m(v_i, w_i)$. Note for two vectors, $a$ and $b$, we say that $a > b$ if $a_k > b_k$ for all $k$; $a \geq b$ if $a_k \geq b_k$ for all $k$ and $a \neq b$; and $a \geq b$ if $a_k \geq b_k$ for all $k$.

The continuity and especially the monotonicity of the member mapping restricts the types of indirect mechanisms. For example, consider an indirect mechanism in which each buyer completes an application form, based on which their membership is decided. This continuous and monotonic direct member mapping is thus valid only when the corresponding composite mapping from indirect application and evaluation processes satisfies the two properties as well.

As is well known from the classical utility theory, the role of the continuity of $m$ is to make the upper contour set closed, and the role of the monotonicity is to make it connected. The monotonicity also implies that $\overline{M}(m)$ is not “thick.”
\( t_k^M(i|\theta) \) is a transfer from buyer \( i \) for good \( k \). We define an \( M \)-mechanism formally.\(^9\)

**Definition 1** A direct mechanism \((q^M, t^M, m, e)\) is an \( M \)-mechanism if for all \( i \in I \), there exists a continuous and monotonic function \( m : \Theta_i \rightarrow \mathbb{R} \) such that for each \( \theta \in \Theta \),

\[
(2) \quad q_M^M(i|\theta) = q_B^M(i|\theta) = 0 \text{ for all } \theta_i \in \Theta_i \setminus M(m).
\]

Let \( \theta_{-i} \) be a vector of all buyers’ types except for buyer \( i \)’s, as an element of \( \Theta_{-i} \), and, similarly, \( v_{-i} \) and \( w_{-i} \) be defined, and additionally, let \( F_{v_{-i}}(v_{-i}) = \times_{j \in \Theta \setminus \{i\}} F_A(k_j) \) and \( F_{w_{-i}}(w_{-i}) = \times_{j \in \Theta \setminus \{i\}} F_B(k_j) \). The expected probabilities that buyer \( i \) obtains good \( A \) and \( B \), respectively, are defined as

\[
Q_A^M(\theta_i) = \int_{\Theta_{-i}} q_A^M(i|\theta) dF_{v_{-i}}(v_{-i}) \times F_{w_{-i}}(w_{-i}) \quad \text{and} \quad Q_B^M(\theta_i) = \int_{\Theta_{-i}} q_B^M(i|\theta) dF_{v_{-i}}(v_{-i}) \times F_{w_{-i}}(w_{-i}).
\]

With the membership fee \( e \), \( t_i^M \) denotes only a payment for purchase. The expected payment that member \( i \) makes to the seller for good \( k \) is

\[
T_k^M(\theta_i) = \int_{\Theta_{-i}} t_k^M(i|\theta) dF_{v_{-i}}(v_{-i}) \times F_{w_{-i}}(w_{-i})
\]

and denote \( T^M(\theta_i) = T_A^M(\theta_i) + T_B^M(\theta_i) \).

Buyer \( i \)’s interim expected payoff conditional on his type if he is a member is

\[
(3) \quad u(\theta_i) = v_i Q_A^M(\theta_i) + w_i Q_B^M(\theta_i) - T^M(\theta_i) - e,
\]

and \( u(\theta_i) = 0 \) if he is not. An \( M \)-mechanism is said to be incentive compatible if \( \forall \theta_i, \theta_i' \in M(m), \)

\[
(4) \quad v_i Q_A^M(\theta_i) - T_A^M(\theta_i) \geq v_i Q_A^M(\theta_i') - T_A^M(\theta_i') \quad \text{and} \quad w_i Q_B^M(\theta_i) - T_B^M(\theta_i) \geq w_i Q_B^M(\theta_i') - T_B^M(\theta_i'),
\]

and

\[
(5) \quad \forall \theta_i \in \Theta_i \setminus M(m), \forall \theta_i' \in M(m), 0 \geq v_i Q_A^M(\theta_i') + w_i Q_B^M(\theta_i') - T^M(\theta_i') - e.
\]

The first condition above is a member’s incentive compatibility such that he does not have an incentive to misreport the other member’s type. His incentive not to “pretend” to be a non-member is implied by the individual rationality condition below. The second condition is a non-member’s incentive compatibility such that he does not have an incentive to pretend

\(^9\)Alternatively, define a set of members such that \( I^M(\theta) \equiv \{i \in I : \theta_i \in M(m)\} \), and a set of “effective” outcomes, \( X^M(\theta) \equiv \{0\} \cup I^M(\theta) \times \{0\} \cup I^M(\theta) \). Then, a mechanism is said to be an \( M \)-mechanism if for all \( i \in I \), there exists a continuous and monotonic function \( m : \Theta_i \rightarrow \mathbb{R} \), identical to all \( i \), such that for each \( \theta \in \Theta \), the sum of an \( M \)-mechanism’s probability allocations \( q^M(x|\theta) \) for the effective outcomes is 1, i.e.,

\[
\sum_{x \in X^M(\theta)} q^M(x|\theta) = 1.
\]

The seller could assign a zero allocation to a member, but he always assigns it to a non-member.
to be a member; his incentive not to misreport the other non-member’s type is trivially satisfied.\textsuperscript{10} In addition, it is said to be individually rational if

\[(6) \quad \forall \theta_i \in M(m), u(\theta_i) \geq 0.\]

Note the separate incentive compatibility in (4) in order for it to represent an indirect membership with separate purchase in which buyers bid for the two goods separately, for example, two second-price auctions for members.

Only members pay the fee $e$ and price $t^M$ for goods, with the zero allocation to a non-member (2), so a non-member’s individual rationality condition can be ignored. Importantly, a non-member’s incentive compatibility (5), i.e., no incentive to lie to be a member, can be simplified to a condition below, with Lemma 1. The condition says that all types in the member standard line must have zero payoff such that

\[(7) \quad \forall \theta_i \in \overline{M}(m), u(\theta_i) = 0.\]

Suppose an “effective” member set; that is, $\Theta_i \setminus M(m) \neq \emptyset$. The incentive compatibility and individual rationality of an $M$-mechanism can be replaced by the member’s incentive compatibility and the “boundary condition,” while the zero payoff above on the member standard line implies the member’s individual rationality.

**Lemma 1** Suppose $\Theta_i \setminus M(m) \neq \emptyset$. Then, an $M$-mechanism $(q^M, t^M, m, e)$ is incentive compatible and individually rational if and only if it satisfies (4) and (7).

Note that an $M$-mechanism’s mapping $m(\cdot)$ and its membership fee $e$ can be chosen such that the member set is the same as the entire type set, $M(m) = \Theta_i$ (e.g., $m(\theta_i) = a$ for any constant $a$, and $e = 0$). Then, given the buyer’s separate participation, the membership becomes the separate selling by Myerson (1981). The definition of an $M$-mechanism is general to include the standard approach, but, in what follows, a membership refers to an effective $M$-mechanism such that $\Theta_i \setminus M(m) \neq \emptyset$.

Lemma 1, on the other hand, poses a challenge to the existence of a member set given the consequence:

\[(8) \quad M(m) = \{\theta_i \in \Theta_i : u(\theta_i) \geq 0\}.\]

\textsuperscript{10}In an indirect mechanism, the seller can implement the latter condition by making a buyer with such type optimally not become a member, not participating in either good’s sales.
The allocation rule $q^M$ and the transfer rule $t^M$ may depend on a set of member types $M(m)$, which yields an interim payoff $u(\theta_i)$. This, combined with Lemma 1, in turn results in the equality $M(m) = \{ \theta_i \in \Theta_i : u(\theta_i) \geq 0 \}$. Hence, the analysis, in nature, involves a fixed point type argument: a member set generates a mechanism, which should yield the same set.

To analyze the mechanism for each dimension, it is convenient to represent the member standard line $M(m)$ as the following two lines.

$$v_e(w_i) \equiv \min\{v_i \in [v, \bar{v}] : \theta_i \in M(m)\}, \quad w_e(v_i) \equiv \min\{w_i \in [w, \bar{w}] : \theta_i \in M(m)\}.$$  

Given a fixed dimension, either $v_e(w_i)$ or $w_e(v_i)$ is the other dimension that is minimally qualified to be a member, which can be “active,” greater than a minimum value, $v$ or $w$, or not, allowing all in the dimension to be qualified. It is the weak monotonicity that demands two separate representations; otherwise, it will be well defined with an inverse function. We can find corresponding supremums $\hat{v}_e \equiv \sup\{v_i \in [v, \bar{v}] : w_e(v_i) > w\}$ and $\hat{w}_e \equiv \sup\{w_i \in [w, \bar{w}] : v_e(w_i) > v\}$, satisfying $w_e(\hat{v}_e) = \bar{w}$ and $v_e(\hat{w}_e) = \bar{v}$. With them, (7) can be restated as $\forall v_i \in [v, \hat{v}_e], u(v_i, w_e(v_i)) = 0$ and $\forall w_i \in [w, \hat{w}_e], u(v_e(w_i), w_i) = 0$.

### 4 Separability and independence of the membership

We say that an $M$-mechanism allocates two goods separably and independently on $M(m)$ if an interim allocation for a good depends only on that good’s valuations on $M(m)$. In one good’s allocation, the other good’s valuation can essentially be treated as non-contractible information, found in Yoo (2016).

**Proposition 1** An incentive compatible and individually rational $M$-mechanism $(q^M, t^M, m, e)$ allocates two goods separably and independently on $M(m)$ such that $\forall (v_i, w_i) \neq (v'_i, w'_i) \in M(m), Q^M_A(v_i, w_i) = Q^M_A(v'_i, w_i)$ for almost all $v_i$ and $\forall (v_i, w_i) \neq (v'_i, w'_i) \in M(m), Q^M_B(v_i, w_i) = Q^M_B(v'_i, w_i)$ for almost all $w_i$.

The separable and independent allocation on $M(m)$ is not surprising; it is based on two separably incentive compatibility conditions in (4). The main focus of this section is rather on the separability and independence of the payment rule on $M(m)$, which is not implied by Proposition 1. The following example illustrates its issues.

**Example 1** (two bidders, symmetric uniform distributions and simple $M$) Consider two buyers $I \equiv \{1, 2\}$, a symmetric and uniform case such that $F_A = F_B = U[0, 1]$. Choose
$m(\theta_i) = \max\{v_i, w_i\} - \frac{1}{2}$, and adopt the second price auction mechanism for each good.

With the symmetry, just examine good A’s allocation. For buyer 1,

$$q_A(v_1, w_1, v_2, w_2) = \begin{cases} 
1 & \text{if } \max\{v_1, w_1\} > \frac{1}{2}, v_1 > v_2,
0 & \text{otherwise},
\end{cases}$$

and

$$t_A(v_1, w_1, v_2, w_2) = \begin{cases} 
v_2 & \text{if } \max\{v_1, w_1\} > \frac{1}{2}, v_1 > v_2, \max\{v_2, w_2\} > \frac{1}{2},
0 & \text{otherwise}.
\end{cases}$$

Buyer 1’s interim payment for good A depends on where his type is: if $v_1 < 1/2, w_1 > 1/2$, it is $\Pr(m(\theta_2) \geq 0, v_2 < v_1)\mathbb{E}[v_2|m(\theta_2) \geq 0, v_2 < v_1] = \int_0^{v_1} \int_0^{1/2} v_2 dv_2 dv_1 = \frac{1}{4}v_1^2$, while if $v_1 > 1/2$, $\Pr(m(\theta_2) \geq 0, v_2 < v_1)\mathbb{E}[v_2|m(\theta_2) \geq 0, v_2 < v_1] = \int_{1/2}^{1} \int_{1/2}^{1} v_2 dv_2 dv_1 + \int_{0}^{v_1} \int_{1/2}^{1} v_2 dv_2 dv_1 = \frac{1}{2}v_1^2 - \frac{1}{16}$. It is not difficult to notice that with the membership, the second price auction mechanism does not satisfy dominant incentive compatibility with a strictly positive membership fee $e > 0$. Furthermore, the above shows that it does not satisfy Bayesian incentive compatibility either: if $v_1 < \frac{1}{2}, w_1 > \frac{1}{2}$, the interim payment is $\frac{1}{4}v_1^2$: buyer 1 has an over-reporting incentive.\footnote{Specifically, given a true type $v_1$ and a report $v_1'$, buyer 1’s interim payoff from good A’s purchase is $v_1v_1' - \frac{1}{4}v_1'^2$, if $v_1 < \frac{1}{2}, w_1 > \frac{1}{2}$.}

We modify the previous $t_A$ such that

$$t_A(v_1, w_1, v_2, w_2) = \begin{cases} 
v_2 & \text{if } \max\{v_1, w_1\} > \frac{1}{2}, v_1 > v_2, v_2 > \frac{1}{2},
2v_2 & \text{if } \max\{v_1, w_1\} > \frac{1}{2}, v_1 > v_2, v_2 < \frac{1}{2},
0 & \text{otherwise}.
\end{cases}$$

Then, regardless of where buyer 1’s type is, his interim payment is the same as $\frac{1}{2}v_1^2$. Such “normalization” retains the second price auction’s Bayesian incentive compatibility, but there is no answer as to why this normalization is necessary at this point, which is a consequence of the next Theorem.

We start by defining type $\theta_i$ member’s payoff from good A, economizing the notations $Q_A^M(v_i) = Q_A^M(v_i, w_i) = Q_A^M(v_i, w_i')$ for any $w_i \neq w_i'$ from Proposition 1, as

$$u_A(v_i, w_i) \equiv v_i Q_A^M(v_i) - T_A^M(v_i, w_i).$$

Then, by the envelope theorem, the interim payment for good A can be rewritten as

$$T_A^M(v_i, w_i) = v_i Q_A^M(v_i) - \int_0^{v_i} Q_A^M(x) dx - u_A(v_i, w_i),$$

\footnote{Specifically, given a true type $v_1$ and a report $v_1'$, buyer 1’s interim payoff from good A’s purchase is $v_1v_1' - \frac{1}{4}v_1'^2$, if $v_1 < \frac{1}{2}, w_1 > \frac{1}{2}$.}
where \( u_A(\nu, w_i) = v_e(w_i)Q^M_A(v_e(w_i)) - T^M_A(v_e(w_i), w_i) - \int_{\nu}^{v_e(w_i)} Q^M_A(x)dx \). If \( v_e(w_i) > \nu \), type \((\nu, w_i)\) is not a member, based on the monotonicity of \( m \), so \( u_A(\nu, w_i) = 0 \), but if \( v_e(w_i) = \nu \), the same argument does not apply, which makes it impossible for us to eliminate \( u_A(\nu, w_i) \).

This difference in (11) can be shown explicitly such that

\[
(12) \quad T^M_A(v_i, w_i) = \begin{cases} 
  v_i Q^M_A(v_i) - \int_{\nu}^{v_i} Q^M_A(x)dx & \text{if } w_i < \widehat{\nu}_e, \\
  v_i Q^M_A(v_i) - \int_{\nu}^{v_i} Q^M_A(x)dx + T^M_A(v, w_i) - v Q^M_A(\nu) & \text{if } w_i \geq \widehat{\nu}_e,
\end{cases}
\]

where, from the definitions \( v_e(w_i) \) and \( w_e(v_i) \) in (9), \( w_i < \widehat{\nu}_e \) is equivalent to \( v_e(w_i) > \nu \), and, similarly, \( w_i \geq \widehat{\nu}_e \) is \( v_e(w_i) = \nu \). An identical procedure applies to the interim payment for good \( B \), \( T^M_B \).

In the single dimension of Myerson (1981), for a seller’s revenue maximization, the lowest valuation must obtain the lowest payoff, identical to a buyer’s outside option. In particular, it is convenient to pin down the lowest valuation’s payoff, before a mechanism characterizes an optimal allocation. But, in a multi-dimension, with an \( M \)-mechanism, a type with the lowest valuation in one dimension can have a high valuation in the other dimension; therefore, it is not immediate whether the revenue maximization leads the lowest valuation to the lowest payoff or not. To examine it further, consider \((v_i, w_i)\) satisfying \( w_i \geq \widehat{\nu}_e \) and \( v_i < \widehat{\nu}_e \), which yields that type’s interim payoff (3), applying Proposition 1 and \( T^M_A, T^M_B \), such that

\[
(13) \quad u(v_i, w_i) = \int_{\nu}^{v_i} Q^M_A(x)dx - [T^M_A(\nu, w_i) - \nu Q^M_A(\nu)] + \int_{\nu}^{w_i} Q^M_B(y)dy - \epsilon.
\]

If the buyer with \((v_i, w_i)\) reports \((v_i, w'_i)\), his payoff is

\[
(14) \quad \int_{\nu}^{v_i} Q^M_A(x)dx - [T^M_A(\nu, w'_i) - \nu Q^M_A(\nu)] + w_i Q^M_B(w'_i) - [w'_i Q^M_B(w'_i) - \int_{\nu}^{w'_i} Q^M_B(y)dy] - \epsilon.
\]

Then for any two \( w_i, w'_i \geq \widehat{\nu}_e \), holding \( v_i < \widehat{\nu}_e \) fixed, a mechanism is incentive compatible if and only if

\[
(15) \quad \int_{\nu}^{w_i} Q^M_B(y)dy - \int_{\nu}^{w'_i} Q^M_B(y)dy \geq w_i Q^M_B(w'_i) - w'_i Q^M_B(w'_i) + T^M_A(\nu, w_i) - T^M_A(\nu, w'_i)
\]

Without the term \( T^M_A(\nu, w_i) - T^M_A(\nu, w'_i) \), a standard procedure of the single dimension yields the incentive compatibility among \( w_i \), given \( v_i \). However, with the term, in a multi-dimension, the monotonicity of \( Q^M_B \) is not sufficient for the incentive compatibility; in particular, the
monotonicity of $Q_B^M$ can fail in “compensating” for the difference, $T_A^M(\overline{v}, w_i) - T_A^M(\overline{v}, w_i')$, so the incentive compatibility interferes with $T_A^M(\overline{v}, w_i) \neq T_A^M(\overline{v}, w_i')$; or it can succeed in compensating.

To show that the incentive compatibility, in fact, implies $T_A^M(\overline{v}, w_i) = T_A^M(\overline{v}, w_i')$ for $w_i \neq w_i'$, and thereby implies that the additional term $T_A^M(\overline{v}, w_i) - \nu Q^M_B(\overline{v})$ can be treated as 0, even when $w_i \geq \widehat{w}_e$ in (12), we must rely on an infinitesimal change of $Q^M_B$, i.e., its differentiability. Thus, we first characterize an optimal allocation rule before we pin down the term $T_A^M(\overline{v}, w_i)$, unlike the single dimension. From the interim payment of (12), the expected revenue from buyer $i$ regarding good $A$’s sale is given as

\[
(16) \quad \int_{M(m)} T_A^M(v_i, w_i) dF_A(v_i) \times F_B(w_i) = \int_{M(m)} \left[ v_i Q_A^M(v_i) - \int_{\underline{v}}^{v_i} Q_A^M(v) dv \right] dF_A(v_i) \times F_B(w_i) \\
+ \int_{M(m)} \left[ T_A^M(v_i, w_i) - Q_A^M_B(\overline{v}) \right] \mathbf{1}_{(w_i \geq \widehat{w}_e)} dF_A(v_i) \times F_B(w_i).
\]

Note that the second term does not depend on the allocation rule, so we focus on the first term. First, by Fubini’s theorem, find the probability that $v_i$ is smaller than $x$, conditional on the member set $M(m)$, denoted by $H^M_A(x) \equiv \Pr(v_i \leq x | M(m))$, such that

\[
(17) \quad H^M_A(x) = \int_{M(m):v_i \leq x} dF_A(v_i) \times F_B(w_i) \\
= \begin{cases} 
\int_{\underline{v}}^{x} \left[ 1 - F_B(w_e(v_i)) \right] dF_A(v_i) & \text{if } x < \widehat{v}_e, \\
\int_{\widehat{v}_e}^{x} \left[ 1 - F_B(w_e(v_i)) \right] dF_A(v_i) + F_A(x) - F_A(\widehat{v}_e) & \text{if } x \geq \widehat{v}_e,
\end{cases}
\]

where $H^M_A$ is a cumulative distribution function with its measure $H^M_A(\overline{v}) < 1$ if the member set excludes a measurable set of types, and $h^M_A$ is its density function, implying differentiability of $H^M_A(x)$ or $H^M_B(y)$ in the two intervals, respectively.\footnote{Note that $w_e(v_i)$ and $v_e(w_i)$ might not be continuous, as in the case of example 1, but if either $w_e(v_i)$ or $v_e(w_i)$ is discontinuous, it is discontinuous only at the break point, either $v_e$ or $w_e$; that is, it is continuous, and, in fact, differentiable in the interval, $x < \widehat{v}_e$, and in the other interval, $x \geq \widehat{v}_e$, separably.}

With $H^M_A(x)$, the first term of each buyer’s interim payment from good $A$ in (16) can be rewritten as $\int_{M(m)} \left[ v_i Q^M_A(v_i) - \int_{\underline{v}}^{v_i} Q^M_A(v) dv \right] dF_A(v_i) \times F_B(w_i) = \int_{\underline{v}}^{v_i} \left[ v_i Q^M_A(v_i) - \int_{\underline{v}}^{v_i} Q^M_A(v) dv \right] dH^M_A(v_i)$.

By defining the $M$-mechanism’s virtual valuation function as

\[
(18) \quad \psi^M_A(x) \equiv x - \frac{H^M_A(\overline{v}) - H^M_A(x)}{h^M_A(x)},
\]

a distribution $H^M_k$ for good $k$’s valuations is said to be “regular,” if the modified virtual valuation function in (18) is strictly increasing, as in the single-dimension case. With the
regularity, the first main result, following the lemma below, shows that the optimal mechanism assigns good $A$ to the buyer with the highest valuation for good $A$.

For each $k \in \{A,B\}$, the regularity of a modified cumulative distribution $H^M_k$ is satisfied for a large class of distributions of $F_k$

**Lemma 2** For each $k \in \{A,B\}$, if $F_k$ is convex, $H^M_k$ is regular for any $M(m)$. If $F_A$ is not convex, then $H^M_A$ is regular under the condition $\int_x [1 - 2F_B(w_e(x)) + F_B(w_e(v_i))] f_A(v_i) dv_i \geq 0$ for all $x$. Similarly, if $F_B$ is not convex, then $H^M_B$ is regular under $\int_y [1 - 2F_A(v_e(y)) + F_A(v_e(w_i))] f_B(w_i) dw_i \geq 0$ for all $y$.

The first main result shows that the membership does not change the optimal relative rule in a revenue-maximizing mechanism: the optimal mechanism allocates good $A$ ($B$) to the member with the highest valuation for that good, resulting in winning probabilities for good $A$ and $B$, $G_A(v_i) \equiv F_A(v_i)^{N-1}$ and $G_B(v_i) \equiv F_B(w_i)^{N-1}$, respectively. The optimal allocation implies more than the independence from Proposition 1; it says that good $A$’s allocation to buyer $i$ depends neither on buyer $i$’s valuation on the other good $B$, $w_i$, nor on the other buyers’ valuations on the other good, $w_{-i}$.

**Theorem 1** Suppose $H^M_k$ is regular for all $k \in \{A,B\}$. An incentive compatible and individually rational $M$-mechanism’s revenue-maximizing optimal allocation for members $\theta_i, \theta_j \in M(m)$ is

$$q^M_A(i|\theta) = \begin{cases} 1 & \text{if } v_i > v_j \text{ for all } j \in I \text{ and } j \neq i, \\ 0 & \text{otherwise}; \end{cases}$$

$$q^M_B(i|\theta) = \begin{cases} 1 & \text{if } w_i > w_j \text{ for all } j \in I \text{ and } j \neq i, \\ 0 & \text{otherwise}. \end{cases}$$

Moreover, the monotonicity of $m(\theta_i)$ implies that for any two member sets $M \neq M'$, $Q^M_A(v_i) = Q^{M'}_A(v_i) = G_A(v_i)$ and $Q^M_B(w_i) = Q^{M'}_B(w_i) = G_B(w_i)$.

The potentially complex existence problem of a member set, based on a fixed point type argument, is resolved from the non-relevance of the member set, for each $M \neq M'$.
\[ Q_A^M(v_i) = Q_A^{M'}(v_i) = G_A(v_i) \quad \text{and} \quad Q_B^M(w_i) = Q_B^{M'}(w_i) = G_B(w_i). \]

Now, we re-examine the additional term \( T_A^M(v, w_i) - vQ_A^M(v) \) in (16). First, by Theorem 1, \( Q_A^M(v) = 0 \), and, in addition, if the buyer with \((v_i, w_i)\) reports \((v_i, w'_i)\) from (14), the differentiability implies now \( T_A^M(v, w_i) = T_A^M(v, w'_i) \) for any pair \((v_i, w_i), (v_i, w'_i) \in M(m)\). And this constant payment \( T_A^M(v, w_i) \), not depending on \( w_i \), can be transferred to a part of the membership fee, without loss of generality. In this way, we solve the two problems with multidimensional analysis, integrability and arbitrary rent – the payoff of each dimension’s lowest type in our model – suggested by Rochet and Choné (1998).

**Corollary 1** An incentive compatible and individually rational \( M \)-mechanism makes transfers for two goods separably and independently on \( M(m) \) such that \( \forall(v_i, w_i) \neq (v_i, w'_i) \),\( (v_i, w'_i) \in M(m), T_A^M(v_i, w_i) = T_A^M(v_i, w'_i) \) for almost all \( v_i \) and \( \forall(v_i, w_i) \neq (v_i', w_i) \),\( (v_i', w_i) \in M(m), T_B^M(v_i, w_i) = T_B^M(v_i', w_i) \) for almost all \( w_i \).

To illustrate the role of the normalization further, in the following example, consider a more general case, an arbitrary membership set.

**Example 2** (two bidders, general distributions and arbitrary \( M \)) Consider two buyers \( I \equiv \{1, 2\} \), a general case with \( F_A, F_B \). Adopt the second price auction mechanism for the member allocation. Bidder 1’s expected payment for good A is \( \Pr(v_e(W_2) < V_2 < v_1)E[V_2|v_e(W_2) < V_2 < v_1] = \int_{V_2}^{v_1} \int_{w_e(v_2)}^{w_2} [v_2f_A(v_2)f_B(v_2)] dw_2 dv_2 = \int_{V_2}^{v_1} v_2[1-F_B(w_e(v_2))]f_A(v_2) dv_2. \)

Similarly, bidder 1’s expected payment for good B is \( \Pr(w_e(V_2) < W_2 < w_1)E[W_2|w_e(V_2) < W_2 < w_1] = \int_{w_2}^{w_1} w_2[1-F_A(w_e(w_2))]f_B(w_2) dw_2. \) If bidder 1 with type \((v_1, w_1) \in M(m)\) reports \((v'_1, w'_1) \in M(m)\), his payoff is

\[
v_1F_A(v'_1) + w_1F_B(w'_1) - \int_{V_2}^{v'_1} v_2[1-F_B(w_e(v_2))]+f_A(v_2) dv_2 - \int_{w_2}^{w'_1} w_2[1-F_A(w_e(w_2))]f_B(w_2) dw_2 - e.
\]

If bidder 1 reports the true valuation, \( v'_1 = v_1 \), the first order derivative with respect to good A’s valuation yields
\[
v_1f_A(v'_1) - v'_1[1-F_B(w_e(v'_1))]f_A(v'_1) > 0 \quad \text{if} \quad w_e(v'_1) > w_e, \quad \text{as in Example 1}.
\]

This implies that the bidder has an incentive to over-report the valuation, greater than the true value \( v_1 \). For the second price to satisfy Theorem 1, its payment rule has to be modified as
\[
t_1 = \frac{1}{[1-F_B(w_e(v_2))]}v_2 \quad \text{if} \quad v_2 < \hat{v}_e.
\]

The Theorem and its corollary enable us to simplify the incentive compatibility (4) and the boundary condition (7) such that \( \forall \theta_i, (v_i, w'_i), (v'_i, w_i) \in M(m) \)

19. \( v_iQ_A^M(v_i) - T_A^M(v_i) \geq v_iQ_A^M(v'_i) - T_A^M(v'_i), \)

20. \( w_iQ_B^M(w_i) - T_B^M(w_i) \geq w_iQ_B^M(w'_i) - T_B^M(w'_i), \)
and

\[ v_i Q^M_A(v_i) - T^M_A(v_i) + w_i Q^M_B(w_i) - T^M_B(w_i) - e = 0. \]

With the optimal mechanism, the member criterion changes only when the membership fee \( e \) changes, so the member set can be rewritten in terms of the fee \( e \) such as \( M(e) \equiv \{ \theta_i \in \Theta_i : v_i Q^M_A(v_i) - T^M_A(v_i) + w_i Q^M_B(w_i) - T^M_B(w_i) \geq e \} \), instead of the entire mapping \( m, M(m) \).

In essence, Theorem 1 reduces the problem of choosing a member set and a membership fee of an \( M \)-mechanism into choosing a membership fee, considering the incentive compatibility condition.

**Corollary 2** An incentive compatible and individually rational \( M \)-mechanism’s optimal allocation and payment rule yield the member criterion \( m(\theta_i) \) and the member set \( M \) such that
\[ m(\theta_i) = \int_{v_i}^{v_i} G_A(x)dx + \int_{w_i}^{w_i} G_B(y)dy - e, \]
and
\[ M(e) = \{ \theta_i \in \Theta_i : \int_{v_i}^{v_i} G_A(x)dx + \int_{w_i}^{w_i} G_B(y)dy \geq e \} \]

We find the revenue-maximizing \( M \)-mechanism given the member set (22) for the comparison between the membership and the separate selling.

## 5 Dominance of the membership

We provide conditions under which the membership dominates separate selling of the two goods. Theorem 1 and Corollary 1, with the member set (22) based on them, yield an \( M \)-mechanism’s maximum revenue from a buyer given each membership fee \( e \) such that

\[ R_M(e) \equiv \int_{M(e)} [T_A(v_i) + T_B(w_i) + e] dF_A(v_i) \times F_B(w_i), \]

where \( T_A(v_i) = \int_{v_i}^{v_i} x dG_A(x) \) and \( T_B(w_i) = \int_{w_i}^{w_i} y dG_B(y) \). On the other hand, the maximum revenue from the separate selling by Myerson (1981) is

\[ R_A(r_A) + R_B(r_B), \]

where \( R_A(r_A) \equiv r_A(1 - F_A(r_A)) G_A(r_A) + \int_{r_A}^{r_A} x dG_A(x) dF_A(v_i) \) and \( R_B(r_B) \equiv r_B(1 - F_B(r_B)) G_B(r_B) + \int_{r_B}^{r_B} y dG_B(y) dF_B(w_i) \), with \( r_A \) and \( r_B \) optimal reserve prices, respectively.

14
In general, finding a closed form solution for the $M$-mechanism’s revenue maximization, its revenue maximizing fee $e$, is intractable. Yet, a simple environment can enable us to do so. For example, revisit example 2 with a symmetric and uniform case such as $F = F_A = F_B = U[0, 1]$.

**Example 3** (two bidders, symmetric uniform distributions) With the symmetry, we suppress the subscripts $k$ in $F_k$ and $H_k$. From (22), the member set is given as $M(e) = \{(v_i, w_i) \in [0, 1]^2 : \frac{1}{2}v_i^2 + \frac{1}{2}w_i^2 - e \geq 0\}$, so the expected revenue from a buyer is $R_M(e) = \int_{M(e)}[T_A(v_i) + T_B(w_i) + e]dF(v_i) \times F(w_i) = \int_0^1 2\psi(x)hx(x)dx + eH^M(1)$, where Theorem 1 shows $Q(x) = x$. Then, $R_M(e) = \int_0^1 (4x^2 - 2x)dx + e - \int_0^{\sqrt{2e}}(x^2 + e)\sqrt{2e - x^2}dx = \int_0^1 (4x^2 - 2x)dx + e - \left(\frac{3\pi}{4}\right)e^2$. The optimal membership fee $e^* = \frac{2}{3\pi}$, and the maximum expected revenue from a buyer is $\int_0^1 (4x^2 - 2x)dx + \frac{1}{3\pi}$. With the optimal fee, as describe in figure 2, we find that $\frac{1}{2} < \sqrt{2e^*}$ and $\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 > 2e^*$, where $\frac{1}{2}$ is the optimal reserve price for the separate selling. The separate selling for two goods yields $R_A(r^*) + R_B(r^*) = 2 \int_{1/\sqrt{2}}^{1/\sqrt{2}} (2z^2 - z)dz$. Compare them to show the dominance of the membership such that $R_M(e^*) - [R_A(r^*) + R_B(r^*)] = \int_0^1 (4x^2 - 2x)dx + \frac{1}{3\pi} - \int_{1/\sqrt{2}}^{1/\sqrt{2}} (4z^2 - 2z)dz = \int_0^{1/\sqrt{2}} (4x^2 - 2x)dx + \frac{1}{3\pi} = -\frac{1}{12} + \frac{1}{3\pi} > 0$. The membership yield a higher revenue.

The approach in Example 3 is not applicable to general cases, including non-uniform distributions. Instead, we connect the membership with separate selling of the two goods, by choosing the membership fee such that one of the member standard line’s intercept from (22) is the same as an optimal reserve price. Now, suppose, WLOG, $\int_a^t G_A(x)dx \geq \int_a^t G_B(y)dy$, as found from figure 3, and choose the membership fee $e = \tilde{c}(r_B)$ such that the member line’s
vertical intercept is the same as good $B$’s optimal reserve price;\footnote{Note that, unlike the uniform example in figure 2, a \textit{general} member line in figure 3 here and figure 4 below does not necessarily have a circle shape: it can be any decreasing function.}

\begin{equation}
\hat{e}(r_B) \equiv \int_{r_B}^{r_B} G_B(x)dx.
\end{equation}

The separate selling yields the maximum revenue from good $A$ such that

\begin{equation}
R_A(r_A) = r_A(1 - F_A(r_A))G_A(r_A) + \int_{r_A}^{v_i} \left[ \int_{r_A}^{v_i} x dG_A(x) \right] dF_A(v_i).
\end{equation}

The ex-ante expected payment for good $A$ from a buyer considers the buyer’s valuations greater than the reserve price $r_A$. Then, the buyer pays $r_A$, if all other buyers’ valuations are smaller than $r_A$, and the buyer pays the second highest valuation conditional on his valuation being the highest otherwise. The first term in the formula above refers to the former and the second term to the latter.

From the membership’s maximum revenue given the fee $\hat{e}(r_B)$, $R_M(\hat{e}(r_B))$ in (23), the membership yields good $A$’s payment from a buyer such that

\begin{equation}
\int_{M(\hat{e}(r_B))} [T_A(v_i)] dF_A(v_i) \times F_B(w_i)
\end{equation}

\[= \int_{r_A}^{r_A} \int_{w_e(v_i)}^{v_i} \left[ \int_{w_e(v_i)}^{v_i} x dG_A(x) \right] dF_B(w_i) dF_A(v_i) + \int_{r_A}^{v_i} \left[ \int_{w_e(v_i)}^{v_i} x dG_A(x) \right] dF_A(v_i).\]
and the second term is to accumulate good A’s valuations of member types whose valuations for good A are greater than that. Note that with a membership, a bidder’s interim payment comes with no reserve price, i.e., $\int_{w}^{v} xdG_{A}(x)$, unlike with the separate selling.\(^{15}\) Hence, expanding the participating type set being positive, the seller cannot prevent buyers from bidding low in the absence of a reserve price. The tradeoff becomes apparent from comparison with the separate selling’s revenue.

The difference in good A’s payment between the membership (27) and the separate selling (26) is

$$\Delta R_{A}(r_{A}) \equiv \int_{x}^{r_{A}} (1 - F_{B}(w_{e}(v_{i}))) \left[ \int_{v_{i}}^{v} xdG_{A}(x) \right] dF_{A}(v_{i}) + (1 - F_{A}(r_{A})) \int_{x}^{r_{A}} xdG_{A}(x)$$

$$- (1 - F_{A}(r_{A})) r_{A} G_{A}(r_{A}),$$

where the above (27) can be rewritten as $\int_{x}^{r_{A}} (1 - F_{B}(w_{e}(v_{i}))) \left[ \int_{v_{i}}^{v} xdG_{A}(x) \right] dF_{A}(v_{i}) + \int_{r_{A}}^{r} \left[ \int_{r_{A}}^{v} xdG_{A}(x) \right] dF_{A}(v_{i}) + \int_{r_{A}}^{r} \left[ \int_{r_{A}}^{v} xdG_{A}(x) \right] dF_{A}(v_{i})$. The first two terms represent revenue gain from a buyer’s payment for good A with the membership, and the last term is revenue loss from no reserve price aggregation that could otherwise come with the separate selling. In return for the additional revenue source, the low bids affect the seller negatively compared with the reserve price from the separate selling. This difference can be succinctly expressed as

$$\int_{x}^{r_{A}} (1 - F_{B}(w_{e}(v_{i}))) \left[ \int_{v_{i}}^{v} xdG_{A}(x) \right] dF_{A}(v_{i}) - (1 - F_{A}(r_{A})) \int_{x}^{r_{A}} G_{A}(x)dx ,$$

where $\int_{x}^{r_{A}} G_{A}(x)dx = r_{A} G_{A}(r_{A}) - \int_{x}^{r_{A}} xdG_{A}(x)$. This will be a key formula for the comparison between the membership and the separate selling.

5.1 Symmetric case

We study the case in which, given the reserve prices from the separate selling, $r_{A}$ and $r_{B}$, $\int_{x}^{r_{A}} G_{A}(x)dx = \int_{x}^{r_{B}} G_{B}(y)dy$ for all $N$, as illustrated in figure 4. Clearly, a symmetric distribution, $F_{A} = F_{B}$, implies this case, but it can hold without such symmetry. From

\(^{15}\)The formula for good A’s payment that the membership yields is the same, regardless of whether the member standard line’s horizontal intercept is the same as the optimal reserve price for good A or not, that is, $\int_{x}^{r_{A}} G_{A}(x)dx = \int_{x}^{r_{B}} G_{B}(y)dy$ or $\int_{x}^{r_{A}} G_{A}(x)dx > \int_{x}^{r_{B}} G_{B}(y)dy$; in other words, $\bar{v}_{e} < r_{A}$ or $\bar{v}_{e} = r_{A}$.  

17
\[ (1 - F_B(w_e(v_i))) \geq (1 - F_B(r_B)) \text{ for } v_i \in [v, r_A], \text{ the key formula (29) satisfies inequality:} \]

\[
\Delta R_A(r_A) \geq \int_{r_A}^{r_B} (1 - F_B(r_B)) \left[ \int_{v_i}^{v} xdG_A(x) \right] dF_A(v_i) - (1 - F_A(r_A)) \int_{v}^{v_i} G_A(x) dx.
\]

We only utilize the area \( I + II + III \) in figure 3 since the remaining area above the member standard line can be tiny given some parameter values; it is impossible to reach a definite answer, contingent on that area.

The above equation (30) can be rewritten as

\[
\int_{v}^{v_i} G_A(x) dx \left[ (1 - F_B(r_B)) D_A(N) + F_A(r_A) F_B(r_B) - 1 \right],
\]

by the integration by parts, \( \int_{v}^{v_i} xdG_A(x) \right] dF_A(v_i) = F_A(r_A) \left[ r_A G_A(r_A) - \int_{v}^{r_A} G_A(x) dx \right] - \int_{v}^{r_A} v_i g_A(v_i) F_A(v_i) dv_i, \) and, in addition, by defining a term such that

\[
D_A(N) \equiv \frac{r_A F_A(r_A) G_A(r_A) - \int_{v}^{r_A} v_i g_A(v_i) F_A(v_i) dv_i}{\int_{v}^{r_A} G_A(v_i) dv_i}.
\]

The following limit result provides a critical step to establish the membership’s dominance in Theorem 2.

**Lemma 3** For each \( k \in \{A, B\}, \lim_{N \to \infty} D_k(N) \geq 1.\)

A similar procedure can be applied to good \( B \) to obtain

\[
\int_{v}^{r_B} G_B(y) dy \left[ (1 - F_A(r_A)) D_B(N) + F_A(r_A) F_B(r_B) - 1 \right].
\]
Additionally, the seller obtains membership fees as a part of the revenue, at least \(1 - F_A(r_A)F_B(r_B)\).

The total of the three revenue sources, with the limit result from Lemma 3, shows that with the symmetric case, for a sufficiently large number of buyers, the revenue from the membership is greater than that from the separate selling.

**Theorem 2** Suppose \(H_k^M\) is regular for all \(k \in \{A,B\}\) and the symmetric case. There exists \(\bar{N}\) such that for all \(N \geq \bar{N}\), the membership dominates the separate selling.

With a sufficiently large number of buyers, the positive effects of the additional revenue source from the membership dominate the negative effects of the low bids without a reserve price. Note that \(R_M(e) - [R_A(r_A) + R_B(r_B)]\) is the revenue difference from a single buyer, so the total difference the seller obtains from the comparison is \(N \times \{R_M(e) - [R_A(r_A) + R_B(r_B)]\}\).

### 5.2 General case

Even if \(\int_\mathbb{R} G_A(x)dx > \int_\mathbb{R} G_B(y)dy\), as in figure 4, the same procedure from the symmetric case applies until the last step. Then, with \(\tilde{c}_N(r_B) = \int_\mathbb{R} G_B(y)dy\),

\[
\int_\mathbb{R} G_A(x)dx \left[ (1 - F_B(r_B))D_A(N) + F_A(r_A)F_B(r_B) - 1 \right]
+ \tilde{c}_N(r_B) \left[ (1 - F_A(r_A))D_B(N) + F_A(r_A)F_B(r_B) - 1 \right] + [1 - F_A(\tilde{v}_e)F_B(r_B)]\tilde{c}_N(r_B).
\]

However, by Lemma 3, for a sufficiently large \(N\), a negative value for \((1 - F_B(r_B))D_A(N) + F_A(r_A)F_B(r_B) - 1\), coupled with the inequality \(\int_\mathbb{R} G_A(x)dx > \int_\mathbb{R} G_B(y)dy\), makes it impossible for us to proceed further to show the dominance.\(^{16}\) Suppose for all \(N \geq \bar{N}\), the following condition is satisfied:

\[
\frac{\int_\mathbb{R} G_A(x)dx}{\int_\mathbb{R} G_B(y)dy} \leq \frac{F_A(\tilde{v}_e)F_B(r_B) - 1}{F_A(r_A)F_B(r_B) - 1}.
\]

It can be shown that for a sufficiently large \(N\), \(\int_\mathbb{R} G_A(x)dx\) and \(\int_\mathbb{R} G_B(y)dy\) are sufficiently close, so the above condition (35) is satisfied.

---

\(^{16}\) That is, for a sufficiently large \(N\), the following reversed inequality impedes the next step:

\[
\int_\mathbb{R} G_A(x)dx \left[ (1 - F_B(r_B))D_A(N) + F_A(r_A)F_B(r_B) - 1 \right] < \tilde{c}_N(r_B) \left[ (1 - F_B(r_B))D_A(N) + F_A(r_A)F_B(r_B) - 1 \right].
\]
Theorem 3 Suppose $H^M_k$ is regular for all $k \in \{A, B\}$. There exists $\overline{N}$ such that for all $N \geq \overline{N}$, the membership dominates the separate selling.

The assumption (35) may require a stronger condition on the number of buyerers, so $\overline{N}$ from the asymmetric case can be different from $\overline{N}$ from the symmetric case earlier.

6 Dominance of the separate selling

For a membership fee $e$, choose a reserve price for good $A$ and a reserve price for good $B$, respectively, such that $r'_A = \hat{v}_e$ and $r'_B = \hat{w}_e$. Consider the key formula in (29) with the stochastic dominance such that

$$\int_{\mathbb{R}} r'_A (1 - F_B(w_e(v_i))) \left[ \int_{\mathbb{R}} xdG_A(x) \right] dF_A(v_i) - (1 - F_A(r'_A)) \int_{\mathbb{R}} G_A(x) dx.$$

Then, we examine only two terms such that

$$r'_A G_A(r'_A) F_A(r'_A) - (1 - F_A(r'_A)) \int_{\mathbb{R}} G_A(x) dx,$$

and, similarly, for good $B$, $r'_B G_B(r'_B) F_B(r'_B) - (1 - F_B(r'_B)) \int_{\mathbb{R}} G_B(y) dy$. By taking small $n$ such as $F_A(v_i)^n$ and $F_B(w_i)^n$ to see the role of the stochastic dominance, the winning probabilities in Theorem 1 can be rewritten as $G_A(x) = F_A(x)^{n(N-1)}$ and $G_B(y) = F_B(y)^{n(N-1)}$.

The theorem below shows that for a sufficiently large $n$, the revenue from the separate selling is greater than that from the membership.

Theorem 4 There exists $\pi$ such that for all $n \geq \pi$, the separate selling dominates the membership.

If the valuation distributions become sufficiently stochastic dominant, the negative effects of the low bids without a reserve price outweigh the positive effects of the additional revenue source from the membership.

7 Concluding remarks

This paper introduces a membership mechanism. The optimal allocation requires the monotonicity of a modified virtual valuation distribution reflecting a member set, which results in a normalization of the optimal payment rule.
We identify two main contrasting factors governing the membership’s dominance over the separate selling. One is the number of bidders and the other is the degree of the stochastic dominance of the valuation distributions.

No attempt has been made to explicitly configure a collection of the two factors, number of bidders and the stochastic dominance, that make the seller indifferent. It is also of theoretical interest to examine conditions under which the current results can still hold, even without the buyer’s separate participation. A buyer or a procurement version of this seller model is the next task as a companion to this paper.

Appendix: Proofs

Proof of Lemma 1. We establish, under (4), the equivalent relationship between (5) & (6) and (7). Show \( \Rightarrow \). Suppose (7) is not satisfied: there exists \( \theta_i = (v_i, w_i) \in \mathbb{M}(m) \) such that \( u(\theta_i) > 0 \). Consider an \( \epsilon \)-ball about \( \theta_i \), \( B_\epsilon(\theta_i) = \{ x \in \mathbb{R}^2_+ : ||x - \theta_i|| < \epsilon \} \) for \( \epsilon > 0 \). First, given \( \Theta_i \setminus M(m) \neq \emptyset \), there exists \( \theta'_i = (v'_i, w'_i) \) sufficiently close to \( \theta_i \) and \( \theta'_i < \theta_i \).

Formally, there exists a sufficiently small \( \epsilon > 0 \) such that \( B_\epsilon(\theta_i) \cap \{ x \in \mathbb{R}^2_+ : x < \theta_i \} \neq \emptyset \). Suppose, on the contrary, \( B_\epsilon(\theta_i) \cap \{ x \in \mathbb{R}^2_+ : x < \theta_i \} = \emptyset \), implying \( \theta_i = (0,0) \), which, together with the monotonicity of \( m \), leads to \( M(m) = \Theta_i \), contradicting \( \Theta_i \setminus M(m) \neq \emptyset \).

Then, choose \( \theta'_i = (v'_i, w'_i) \in B_\epsilon(\theta_i) \cap \{ x \in \mathbb{R}^2_+ : x < \theta_i \} \). Since \( \theta'_i \) is not a member type, \( u(v'_i, w'_i) < 0 < u(v_i, w_i) \), which is equivalently rewritten as

\[
u(v'_i, w'_i) < v_i Q_A^M(v_i, w_i) + w_i Q_B^M(v_i, w_i) - T^M(v_i, w_i) - e
\]

\[
< v'_i Q_A^M(v_i, w_i) + w'_i Q_B^M(v_i, w_i) - T^M(v_i, w_i) - e,
\]

where the last inequality holds for a sufficiently small \( \epsilon > 0 \), violating the incentive compatibility (5). Now, show \( \Leftarrow \). Given that (7) and the monotonicity of \( m \) imply (6), it remains to show (5). Suppose (5) is not satisfied: there exists \( (v_i, w_i) \in \Theta_i \setminus M(m) \) such that

\[
0 < v_i Q_A^M(v'_i, w'_i) + w_i Q_B^M(v'_i, w'_i) - T^M(v'_i, w'_i) - e
\]

for some \( (v'_i, w'_i) \in M(m) \).

But then, for any \( (v'', w'') \in \mathbb{M}(m) \) such that \( (v'', w'') > (v_i, w_i) \),

\[
u(v'', w'') = 0 < v_i Q_A^M(v_i, w_i) + w_i Q_B^M(v_i, w_i) - T^M(v_i, w_i) - e
\]

\[
\leq v'' Q_A^M(v'_i, w'_i) + w'' Q_B^M(v'_i, w'_i) - T^M(v'_i, w'_i) - e,
\]

which contradicts the incentive compatibility among members (4), even for either \( Q_A^M(v'_i, w'_i) = 0 \) or \( Q_B^M(v'_i, w'_i) = 0 \).
Proof of Proposition 1. We show that for any incentive compatible mechanism on \( M(m) \), the allocation and payment for one good does not depend on the report of the other good’s value. First, examine good \( A \)’s incentive compatibility from (4):

\[
(38) \quad v_i Q_A^M(\theta_i) - T_A^M(\theta_i) \geq v_i Q_A^M(v'_i, w'_i) - T_A^M(v'_i, w'_i).
\]

This is equivalent to \( \forall (v_i, w_i) \neq (v'_i, w'_i) \in M(m), \)

\[
(39) \quad v_i Q_A^M(\theta_i) - T_A^M(\theta_i) \geq v_i Q_A^M(v'_i, w_i) - T_A^M(v'_i, w_i),
\]

\[
(40) \quad v_i Q_A^M(\theta_i) - T_A^M(\theta_i) = v_i Q_A^M(v_i, w'_i) - T_A^M(v_i, w'_i).
\]

If we fix \( v_i \) or fix \( w_i \), it is immediate that the incentive compatibility in (38) implies the incentive compatibility with (39)-(40). On the other hand, by combining (39) with (40), the latter implies the former. In particular, (40) shows that for any mechanism inducing each buyer to report \( w_i \) truthfully for a fixed \( v_i \), the buyer should have an identical payoff.

From the incentive compatibility (39)-(40), for good \( A \), a direct mechanism is incentive compatible on \( M(m) \) if and only if \( Q_A^M \) is increasing in \( v_i ; \forall (v_i, w_i) \in M(m), \)

\[
(41) \quad T_A^M(v_i, w_i)
= v_i Q_A^M(v_i, w_i) + (T_A^M(v_i(w_i), w_i) - v_i(w_i)Q_A^M(v_i(w_i), w_i)) - \int_{v_i(w_i)}^{v_i} Q_A^M(x, w_i)dx;
\]

and \( \forall (v_i, w_i), (v_i, w'_i) \in M(m), \)

\[
v_i Q_A^M(v_i, w_i) - T_A^M(v_i, w_i) = v_i Q_A^M(v_i, w'_i) - T_A^M(v_i, w'_i).
\]

The same procedure applies to good \( B \)’s incentive compatibility. Now, consider \( (v_i, w_i), (v_i, w'_i) \in M(m) \) with \( w_i \neq w'_i \). Define \( D(v_i) \) such that

\[
D(v_i) \equiv v_i(w_i)Q_A^M(v_i(w_i), w_i) - T_A^M(v_i(w_i), w_i) - \int_{v_i(w_i)}^{v_i} Q_A^M(y, w_i)dy
-
\left[ v_i(w'_i)Q_A^M(v_i(w'_i), w'_i) - T_A^M(v_i(w'_i), w'_i) - \int_{v_i(w'_i)}^{v_i} Q_A^M(y, w'_i)dy \right].
\]

Then, from (40) and (41), \( D(v_i) = 0 \) for all \( v_i \). By the mean value theorem and the fundamental theorem of calculus, \( D'(v_i) = 0 \) for almost all \( v_i \). ■

Proof of Lemma 2. Taking the derivative of the virtual valuation yields

\[
\psi_A^M(x) = 1 + \frac{[h_A^M(x)]^2 + h_A^M(x)[H_A^M(\overline{v}) - H_A^M(x)]}{[h_A^M(x)]^2}
\]
where

\[ h^M_A(x) = \begin{cases} 
[1 - F_B(w_e(x))]f_A(x) & \text{if } x \leq \hat{v}_e, \\
 f_A(x) & \text{if } x > \hat{v}_e,
\end{cases} \]

and

\[ h^{M'}_A(x) = \begin{cases} 
-f_B(w_e(x))w'_e(x)f_A(x) + [1 - F_B(w_e(x))]f'_A(x) & \text{if } x \leq \hat{v}_e, \\
f'_A(x) & \text{if } x > \hat{v}_e.
\end{cases} \]

Since \( w'_e(x) \leq 0 \), if \( F_A \) is convex, then \( H_A(x) \) is regular. Suppose \( F_A(x) \) is not convex. Examine the two ranges in (17) separately. First, if \( x > \hat{v}_e \), the modified virtual valuation distribution becomes the standard virtual valuation distribution: \( \psi^M_A(x) = x - \frac{1 - F_A(x)}{f_A(x)} \). Now, consider \( x \leq \hat{v}_k \). For \( x \leq \hat{v}_k \), the numerator of \( \psi^M_A(x) \) is

\[
2[h^M_A(x)]^2 + h^M_A(x)[H^M_A(\tau) - H^M_A(x)]
\]

\[
= 2\left\{ [1 - F_B(w_e(x))]f_A(x) \right\}^2 - \left\{ f_B(w_e(x))w'_e(x)f_A(x) - [1 - F_B(w_e(x))]f'_A(x) \right\}[H^M_A(\tau) - H^M_A(x)]
\]

\[
= 2\left\{ [1 - F_B(w_e(x))]f_A(x) \right\}^2 - f_B(w_e(x))w'_e(x)f_A(x)[H^M_A(\tau) - H^M_A(x)]
+ [1 - F_B(w_e(x))]f'_A(x)[H^M_A(\tau) - H^M_A(x)].
\]

By the monotone hazard rate, we have \( f'_A(x) \geq -\frac{f_A(x)^2}{1 - F_A(x)} \), so the above can be rewritten as

\[
2\left\{ [1 - F_B(w_e(x))]f_A(x) \right\}^2 - f_B(w_e(x))w'_e(x)f_A(x)[H^M_A(\tau) - H^M_A(x)]
+ [1 - F_B(w_e(x))]f'_A(x)[H^M_A(\tau) - H^M_A(x)]
\geq 2\left\{ [1 - F_B(w_e(x))]f_A(x) \right\}^2 - f_B(w_e(x))w'_e(x)f_A(x)[H^M_A(\tau) - H^M_A(x)]
- [1 - F_B(w_e(x))]rac{f_A(x)^2}{1 - F_A(x)}[H^M_A(\tau) - H^M_A(x)].
\]

From \( w'_e(x) \leq 0 \), we only need to consider the first and the third terms, and by combining the two terms, we have

\[
\left\{ [1 - F_B(w_e(x))]f_A(x) \right\}^2 \left\{ 2 - \frac{H^M_A(\tau) - H^M_A(x)}{[1 - F_A(x)][1 - F_B(w_e(x))]} \right\},
\]

where

\[
H^M_A(\tau) - H^M_A(x) = \int_x^{\hat{v}_e} [1 - F_B(w_e(v_i))] f_A(v_i)dv_i + 1 - F_A(\hat{v}_e)
\]

\[
= 1 - F_A(x) - \int_x^{\hat{v}_e} F_B(w_e(v_i))f_A(v_i)dv_i.
\]
The numerator of \( 2 - \frac{H^M_A(x) - H^M_A(v)}{|1 - F_A(x)| |1 - F_B(w_e(x))|} \) is
\[
2[1 - F_A(x)][1 - F_B(w_e(x))] - [H^M_A(v) - H^M_A(x)]
\]
\[
= [1 - F_A(x)][1 - 2F_B(w_e(x))] + \int_{x}^\infty F_B(w_e(v_i)) f_A(v_i) dv_i
\]
\[
= [1 - F_A(x)][1 - 2F_B(w_e(x))] + \int_{x}^\infty F_B(w_e(v_i)) f_A(v_i) dv_i,
\]
where the last equality follows from the fact that \( F_B(w_e(v_i)) = 0 \) for all \( v_i > \widehat{v}_e \). Then,
\[
[1 - F_A(x)][1 - 2F_B(w_e(x))] + \int_{x}^\infty F_B(w_e(v_i)) f_A(v_i) dv_i
\]
\[
= \int_{x}^\infty [1 - 2F_B(w_e(x))] f_A(v_i) dv_i + \int_{x}^\infty F_B(w_e(v_i)) f_A(v_i) dv_i
\]
\[
= \int_{x}^\infty [1 - 2F_B(w_e(x)) + F_B(w_e(v_i))] f_A(v_i) dv_i.
\]
Hence, if \( \int_{x}^\infty [1 - 2F_B(w_e(x)) + F_B(w_e(v_i))] f_A(v_i) dv_i \geq 0, H^M_A(x) \) is regular.

**Proof of Theorem 1.** From Proposition 1, for any \( \theta_i \in M(m) \),
\[
Q^M_A(v_i) = Q^M_A(\theta_i) \equiv \int_{\Theta_{-i}} q^M_A(i|\theta) dF_{v_{-i}}(v_{-i}) \times F_{w_{-i}}(w_{-i}),
\]
so the expected revenue from a buyer for good \( A \) is
\[
\int_{M(m)} \left[ v_i Q^M_A(v_i) - \int_{u}^{v_i} Q^M_A(x) dx \right] dF_{v_{-i}}(v_{-i}) \times F_{w_{-i}}(w_{-i})
\]
\[
= \int_{u}^{v_i} \left[ v_i Q^M_A(v_i) - \int_{u}^{v_i} Q^M_A(x) dx \right] dH_A(v_i)
\]
\[
= \int_{u}^{v_i} Q^M_A(v_i) \psi^M_A(v_i) h_A(v_i) dv_i
\]
\[
= \int_{u}^{v_i} Q^M_A(v_i) [1 - F_B(w_e(v_i))] \psi^M_A(v_i) f_A(v_i) dv_i
\]
\[
= \int_{\Theta_i} Q^M_A(v_i) [1 - F_B(w_e(v_i))] \psi^M_A(v_i) dF_{v_{-i}}(v_{-i}) \times F_{w_{-i}}(w_{-i}),
\]
where the second equality follows from the typical step of a single dimension. Now,
\[
\sum_{i \in I} \left[ \int_{u}^{v_i} Q^M_A(v_i) \psi^M_A(v_i) h_A(v_i) dv_i \right]
\]
\[
= \sum_{i \in I} \left[ \int_{\Theta_i} Q^M_A(v_i) [1 - F_B(w_e(v_i))] \psi^M_A(v_i) dF_{v_{-i}}(v_{-i}) \times F_{w_{-i}}(w_{-i}) \right]
\]
\[
= \sum_{i \in I} \left[ \int_{\Theta_i} q^M_A(i|\theta) [1 - F_B(w_e(v_i))] \psi^M_A(v_i) dF_{v}(v) \times F_{w}(w) \right],
\]
24
where \( F_a(v) \equiv \times_{j \in I} F_A(v_j) \) and, similarly, \( F_w(w) \equiv \times_{j \in I} F_B(w_j) \). A critical difference between this and Myerson (1981) is that we have a term \([1 - F_B(w_e(v_i))]\), in addition to the modified virtual valuation function \( \psi^M_A(v_i) \). If the distribution \( H^M_A \) is regular, then the virtual valuation is strictly increasing, and, furthermore, the additional term \([1 - F_B(w_e(v_i))]\) is also increasing, given that \( w_e(v_i) \) is decreasing. Hence, we can obtain a similar characterization for the optimal allocation as in the single dimension. \( \blacksquare \)

**Proof of Lemma 3.** Consider the numerator of (32). By incorporating \( G_A(x) = F_A(x)^{N-1} \) and \( g_A(x) = (N-1)F_A(x)^{N-2}f_A(x) \), it can be rewritten as

\[
r_AF_A(r_A)G_A(r_A) - \int_{v}^{r_A} v_i g_A(v_i) F_A(v_i) dv_i
\]

\[
= r_AF_A(r_A)^N - r_A \frac{N-1}{N} F_A(r_A)^N + \int_{v}^{r_A} \frac{N-1}{N} F_A(v_i)^N dv_i
\]

\[
= \frac{r_AF_A(r_A)^N}{N} + \frac{N-1}{N} \int_{v}^{r_A} F_A(v_i)^N dv_i,
\]

where, by the integration by parts, \( \int_{v}^{r_A} v_i g_A(v_i) F_A(v_i) dv_i = r_A \frac{N-1}{N} F_A(r_A)^N - \int_{v}^{r_A} \frac{N-1}{N} F_A(v_i)^N dv_i \).

Hence,

\[
\frac{r_AF_A(r_A)G_A(r_A) - \int_{v}^{r_A} v_i g_A(v_i) F_A(v_i) dv_i}{N \int_{v}^{r_A} F_A(v_i)^N dv_i}
\]

\[
= \frac{r_AF_A(r_A)^N + (N-1) \int_{v}^{r_A} F_A(v_i)^N dv_i}{N \int_{v}^{r_A} F_A(v_i)^N dv_i}
\]

\[
= \frac{r_A \int_{v}^{r_A} [N F_A(v_i)^N - f_A(v_i)] dv_i + N \int_{v}^{r_A} F_A(v_i)^N dv_i}{N \int_{v}^{r_A} F_A(v_i)^N dv_i}
\]

\[
= \frac{r_A \int_{v}^{r_A} [N F_A(v_i)^N - f_A(v_i)] dv_i + N \int_{v}^{r_A} F_A(v_i)^N dv_i}{N \int_{v}^{r_A} F_A(v_i)^N dv_i}
\]

First, the second term of the formula above satisfies that for all \( n \geq 2 \),

\[
F_A(v_i)^N < F_A(v_i)^{N-1} < \frac{N}{N-1} F_A(v_i)^{N-1}
\]

so

\[
\frac{\int_{v}^{r_A} F_A(v_i)^N dv_i}{N \int_{v}^{r_A} F_A(v_i)^{N-1} dv_i} < \frac{1}{N-1} \Rightarrow \lim_{{N \to \infty}} \frac{\int_{v}^{r_A} F_A(v_i)^N dv_i}{N \int_{v}^{r_A} F_A(v_i)^{N-1} dv_i} \leq \lim_{{N \to \infty}} \frac{1}{N-1} = 0.
\]
Now, the first term can be rewritten as
\[
\frac{\int_u^r [r_A F_A(v_i)]^{N-1} f_A(v_i) dv_i + \int_u^r F_A(v_i)^N dv_i}{\int_u^r F_A(v_i)^{N-1} dv_i} = \frac{\int_u^r F_A(v_i)^{N-1} [v f_A(v_i) + F_A(v_i)] dv_i}{\int_u^r F_A(v_i)^{N-1} dv_i},
\]
where
\[
\frac{\int_u^r F_A(v_i)^{N-1} [v f_A(v_i) + F_A(v_i)] dv_i}{\int_u^r F_A(v_i)^{N-1} dv_i} > \frac{\int_u^r F_A(v_i)^{N-1} [v f_A(v_i) + F_A(v_i)] dv_i}{\int_u^r F_A(v_i)^{N-1} dv_i} = \frac{\int_u^r F_A(v_i)^{N-1} [v f_A(v_i) - 1 + F_A(v_i) + 1] dv_i}{\int_u^r F_A(v_i)^{N-1} dv_i} = \frac{\int_u^r F_A(v_i)^{N-1} f_A(v_i) \left[v - \frac{1 - F_A(v_i)}{f_A(v_i)}\right] dv_i}{\int_u^r F_A(v_i)^{N-1} dv_i} + 1.
\]

Note that, by the integration by parts,
\[
\int_u^r F_A(v_i)^{N-1} f_A(v_i) \left[v - \frac{1 - F_A(v_i)}{f_A(v_i)}\right] dv_i = \frac{1}{N} F_A(v_i)^N \left[v - \frac{1 - F_A(v_i)}{f_A(v_i)}\right] \bigg|_u^r - \int_u^r \frac{1}{N} F_A(v_i)^N \left[v - \frac{1 - F_A(v_i)}{f_A(v_i)}\right]' dv_i = - \int_u^r \frac{1}{N} F_A(v_i)^N \left[v - \frac{1 - F_A(v_i)}{f_A(v_i)}\right]' dv_i.
\]

Hence, the first term satisfies the following inequality such that
\[
\frac{\int_u^r F_A(v_i)^{N-1} [r_A f_A(v_i) + F_A(v_i)] dv_i}{\int_u^r F_A(v_i)^{N-1} dv_i} > - \frac{\int_u^r \frac{1}{N} F_A(v_i)^N \left[v - \frac{1 - F_A(v_i)}{f_A(v_i)}\right]' dv_i}{N \int_u^r F_A(v_i)^{N-1} dv_i} + 1.
\]

For any \(N \geq 2\),
\[
\left[v - \frac{1 - F_A(v_i)}{f_A(v_i)}\right]' F_A(v_i)^N < \frac{N}{N-1} \left[v - \frac{1 - F_A(v_i)}{f_A(v_i)}\right]' F_A(v_i)^N \leq \frac{N}{N-1} \overline{B} F_A(r_A) F_A(v_i)^{N-1}
\]
where for any \(v_i \in [v, r_A]\), \(\left[v - \frac{1 - F_A(v_i)}{f_A(v_i)}\right]' \leq \overline{B}\) for some \(\overline{B}\). Thus,
\[
\frac{\int_u^r \frac{1}{N} F_A(v_i)^N \left[v - \frac{1 - F_A(v_i)}{f_A(v_i)}\right]' dv_i}{\int_u^r F_A(v_i)^{N-1} dv_i} < \frac{\overline{B} F_A(r_A)}{N-1}
\]

\(\Rightarrow\) \(\lim_{N \to \infty} \frac{\int_u^r \frac{1}{N} F_A(v_i)^N \left[v - \frac{1 - F_A(v_i)}{f_A(v_i)}\right]' dv_i}{\int_u^r F_A(v_i)^{N-1} dv_i} \leq \lim_{N \to \infty} \frac{\overline{B} F_A(r_A)}{N-1} = 0.\)
This completes the proof.

**Proof of Theorem 2.** Similarly, for good $B$, we have
\[
\int_w^{r_B} G_B(y) dy \left[ (1 - F_B(r_A)) D_B(N) + F_A(r_A) F_B(r_B) - 1 \right],
\]
where
\[
D_B(N) \equiv \frac{[r_B G_B(r_B) F_B(r_B) - \int_w^{r_B} w_B(w_i) F_B(w_i) dw_i]}{\int_w^{r_B} G_B(y) dy}.
\]
Now, we have at least the following membership fee $[1 - F_A(r_A) F_B(r_B)] e(r_B)$, where $e(r_B) \equiv \int_w^{r_B} G_B(x) dx$ from (25). Over the sequence, this value $e$ can change, so it is denoted by $e_N(r_B)$. Hence,
\[
\int_w^{r_A} G_A(x) dx \left[ (1 - F_B(r_B)) D_A(N) + F_A(r_A) F_B(r_B) - 1 \right]
+ \int_w^{r_B} G_B(y) dy \left[ (1 - F_A(r_A)) D_B(N) + F_A(r_A) F_B(r_B) - 1 \right]
+ [1 - F_A(r_A) F_B(r_B)] e_N(r_B)
= e_N(r_B) \left[ (1 - F_B(r_B)) D_A(N) + (1 - F_A(r_A)) D_B(N) + F_A(r_A) F_B(r_B) - 1 \right]
\]
where we examine the limit of the part inside of the term
\[
\lim_{N \to \infty} \left[ (1 - F_B(r_B)) D_A(N) + (1 - F_A(r_A)) D_B(N) + F_A(r_A) F_B(r_B) - 1 \right] \geq (1 - F_A(r_A))(1 - F_B(r_B)) > 0,
\]
which establishes the theorem.

**Proof of Theorem 3.** We first show that as $N \to \infty$, $|G_A(x) - G_B(x)| \to 0$ for all $x$.
By the mean value theorem, there exists $k < 1$ such that
\[
|G_A(x) - G_B(x)| = |F_A(x)^{N-1} - F_B(x)^{N-1}| = |F_A(x) - F_B(x)|(N - 1)k^{N-2}.
\]
Furthermore,
\[
\begin{align*}
&\left| \int_w^{r_A} G_A(x) dx - \int_w^{r_B} G_B(y) dy \right| \\
&\leq \left| \int_w^{r_A} G_A(x) dx - \int_w^{r_A} G_A(x) dx \right| + \left| \int_w^{r_A} G_A(x) dx - \int_w^{r_B} G_A(x) dx \right| \\
&\quad + \left| \int_w^{r_B} G_A(x) dx - \int_w^{r_B} G_B(x) dx \right| \\
&= |r_A - r_B| G_A(r_C) + \left| \int_w^{r_B} G_A(x) dx - \int_w^{r_B} G_B(x) dx \right|
\end{align*}
\]
If the condition (35) is satisfied, then (34) is
\[
\int_{\underline{w}}^{r_A} G_A(x)dx \left[ (1 - F_B(r_B)) D_A(N) + F_A(r_A) F_B(r_B) - 1 \right]
\]
\[
+ \hat{e}_N(r_B) \left[ (1 - F_A(r_A)) D_B(N) + F_A(r_A) F_B(r_B) - 1 \right] + \left[1 - F_A(\hat{e}) F_B(r_B)\right] \hat{e}_N(r_B)
\]
\[
\geq \int_{\underline{w}}^{r_A} G_A(x)dx (1 - F_B(r_B)) D_A(N) + \hat{e}_N(r_B) \left[ (1 - F_A(r_A)) D_B(N) + F_A(r_A) F_B(r_B) - 1 \right]
\]
\[
\geq \hat{e}_N(r_B) (1 - F_B(r_B)) D_A(N) + \hat{e}_N(r_B) \left[ (1 - F_A(r_A)) D_B(N) + F_A(r_A) F_B(r_B) - 1 \right],
\]
where the last inequality follows from \( \int_{\underline{w}}^{r_A} G_A(x)dx > \int_{\underbar{w}}^{r_B} G_B(y)dy = \hat{e}_N(r_B). \)

**Proof of Theorem 4.** Fix \( N, \) and consider only good \( A. \) From (37), we have
\[
r_A F_A(r_A)^n - (1 - F_A(r_A)) \int_{\underline{w}}^{r_A} F_A(x)^{n(N-1)}dx
\]
\[
= r_A F_A(r_A)^n N - (1 - F_A(r_A)) \int_{\underline{w}}^{r_A} F_A(x)^{n(N-1)}dx
\]
\[
= \int_{\underline{w}}^{r_A} F_A(x)^{(n-1)}dx \left[ F_A(r_A)^n + \frac{r_A F_A(r_A)^n}{\int_{\underline{w}}^{r_A} F_A(x)^{n(N-1)}dx} - 1 \right],
\]
where
\[
\frac{r_A F_A(r_A)^n}{\int_{\underline{w}}^{r_A} F_A(x)^{n(N-1)}dx} = \frac{r_A F_A(r_A)^n}{\int_{\underline{w}}^{r_A} F_A(x)^{n(N-1)}dx} F_A(r_A)^n.
\]
Using the integration by parts,
\[
\int_{\underline{w}}^{r_A} x \left[ n(N-1) F_A(x)^{n(N-1)-1} f(x) \right] dx = r_A F_A(r_A)^n(N-1) - \int_{\underline{w}}^{r_A} F_A(r_A)^n(N-1)dx,
\]
and
\[
n(N-1) F_A(x)^{n(N-1)-1} = \frac{n(N-1)}{F_A(x)^{-n(N-1)-1}},
\]
so \( \lim_{n \to \infty} n(N-1) F_A(x)^{n(N-1)-1} = 0. \)

**References**


Yoo, S.H. (2016), Mechanism Design with Non-Contractible Information, a working paper.