Discussion Paper Series
No. 1903
November 2020

Matching Strategic Agents on a Two-Sided Platform

Masaki Aoyagi Seung Han Yoo

The Institute of Economic Research – Korea University
Anam-dong, Sungbuk-ku, Seoul, 136-701, South Korea, Tel: (82-2) 3290-1632, Fax: (82-2) 928-4948

Matching Strategic Agents on a Two-Sided Platform*

Masaki Aoyagi† Osaka University

Seung Han Yoo‡ Korea University

November 10, 2020

Abstract

A platform offers sellers and buyers trading opportunities by creating one-to-one matches between them. A matching mechanism consists of a menu of subscription plans for each side and specifies fees and the probabilities with which subscribers of each plan are matched with subscribers of different plans on the other side. We characterize optimal *matching mechanisms* which maximize the subscription revenue under the incentive compatibility conditions. When the agents are strategic in their interactions with their matched partners, we show that the optimal matching rule may not equal socially efficient positive assortative matching (PAM) but instead focus on the extraction of the agents' informational rents. We then examine the efficiency of the optimal mechanism in two alternative scenarios in which the platform exercises stronger control over the interactions between the matched agents. When the subscription fee can be conditioned on the success of a transaction, we show that the optimal mechanism is efficient with PAM restored as the optimal matching rule. However, when the platform has full control over the allocation and price of the good, we show that the optimal mechanism employs PAM but may create efficiency distortions by blocking some efficient transactions.

Key words: assortative, screening, auction, subscription, revenue maximization.
JEL Codes: D42, D47, D62, D82, L12

1 Introduction

Platforms that match agents flourish in modern economies with the development of information technologies. They realize gains from trade and other forms of interactions by providing agents access to each other: an internet auction house matches sellers and buyers who would otherwise not be able to find trading partners, a job matching platform matches firms and workers who

*We are grateful to seminar participants at various universities for their comments. Aoyagi acknowledges financial support from the JSPS (grant numbers: 21653016, 24653048, 15K13006, 22330061, 15H03328, 15H05728, and 20H05631), the Joint Usage/Research Center at ISER, Osaka University, and the International Joint Research Promotion Program of Osaka University. Yoo acknowledges financial support from a Korea University research grant. Part of this research was conducted while Yoo was visiting ISER, Osaka University. The paper was formerly circulated under the title "Matching Platforms".

†aoyagi@iser.osaka-u.ac.jp.

‡shyoo@korea.ac.kr.

would otherwise face under-utilization of their resources, and a crowd-funding platform matches entrepreneurs with investors to create new businesses. While there is now sizable literature on matching platforms, one important aspect of matching platforms yet to be explored concerns the facts that the interactions between their subscribers are often strategic, and that their strategic incentives in such interactions are determined by how a match is formed by the platforms. Sellers of items in a trading platform post prices that are optimal given the expected willingness to pay of subscribing buyers for his goods, and bidders in an auction platform choose bids that are optimal given their beliefs about the valuations of other subscribing bidders. Put differently, subscribers to a platform play a game against each other, and the outcome of their interactions is an equilibrium of the game. In particular, when the subscribers are privately informed about their types, they play incomplete information games, and the value of a match to each of them is endogenously determined by their Bayes Nash equilibrium (BNE) payoffs. Since the way the platform matches subscribers changes their beliefs about their opponents, it also changes their equilibrium behavior and payoffs. Given that the equilibrium payoffs determine the agents' willingness to pay in the subscription stage, the platform attempting to maximize subscription revenue wants to match agents in such a way that they play its preferred BNE. For example, an auction platform may attempt to restrict subscription to high valuation bidders so as to create competition among them and charge a high subscription fee to sellers. The objective of this paper is to study how a matching platform can maximize its subscription revenue by controlling a *matching rule* which determines the subscribers' beliefs in the game played amongst its subscribers. Our approach marks a departure from the literature on a matching platform which assumes that the match value to each subscriber is an exogenous function of their own type as well as those of the matched subscribers.

In our baseline model of *two-sided competition,* agents from the two sides of a market take strategic behavior against each other. Specifically, a trading platform creates one-to-one matches of sellers and buyers where each buyer makes a take-it-or-leave-it offer to the matched seller, who then accepts or rejects the offer. The sellers have two cost types, whereas the buyers have two valuation types. These types are overlapped so that efficient trading is possible within a match when it involves a high-valuation buyer or a low-cost seller. The *matching mechanism* of the platform specifies a menu of *subscription plans* for each side, and specifies fees as well as the probabilities with a subscriber of each plan is matched with subscribers of different plans on the other side. In other words, the matching mechanism is a simultaneous screening device of agents on both sides. Such screening of subscribers based on subscription plans is a common practice. A subscription plan that puts sellers at the top of search outcomes for a higher price is a leading example of such a screening device: The sellers' ranking in the list will affect the probability that they will be matched with buyers with high-willingness to pay.[1]

In our model, each subscription plan is designed for each agent type, and a matching mechanism is incentive compatible if the agents choose the plans designed for them. The optimal mechanism maximizes the platform's subscription revenue under this incentive compatibility requirement. The agents' choice of a subscription plan depends on both the required payment as well as the value of the expected match. Importantly, this value is the BNE payoff of the game they play in the

[1]Other examples include eBay, which offers sellers and buyers the option "eBay plus" in some countries that their raises visibility to the other side of the market, and a matchmaking platform Vidaselect, which offers a premium membership which allows men better access to women.

match, and will depend on the agent's own type as well as their *belief* about the type of the agent they are matched with. The main strategic consideration is embodied in a high-valuation buyer's strategy with the cut-off property in terms of his belief about the seller's type: He will bid low if his belief weight on the low-cost seller is above some threshold, but bid high otherwise. Since a high-valuation buyer's incentive changes with the match probability, so does the subscription revenue of the platform.

Our first observation is that a matching mechanism maximizes social surplus if and only if it entails positive assortative matching (PAM) that matches a high-valuation buyer with a low-cost seller as much as possible. Hence, all matching mechanisms that entail PAM and allow any gains from trade to realize are first-best efficient although they may generate different welfare distributions to the agents and the platform. Our main focus hence is on whether the optimal mechanism has such a property. We show that the matching rule under the optimal mechanism takes one of three forms depending on the proportion of the agents' types in the population. PAM is among them and so is random matching (RM) that generates matches agents according to their type shares in the population. Under the third matching rule referred to as B-squeeze matching (BSM), a high-valuation buyer is matched with a low-cost seller more often than a low-valuation buyer is, but not to the maximal extent as under PAM. It instead squeezes the high-valuation buyers by minimizing their informational rents. BSM matching rule is optimal when there are a high proportion of high-valuation buyers and a medium to high proportion of low-cost sellers.

The suboptimality of PAM for some type distributions is in sharp contrast with the finding in the literature, and implies divergence between welfare maximization and profit maximization. In particular, the optimality of BSM is a direct consequence of the strategic incentive of high-valuation buyers as mentioned above, and showcases itself in the most striking form in a *symmetric* market where the proportion of low-cost buyers is the same as the proportion of high-valuation buyers: In such a market, it is both physically feasible and socially efficient to match a high-valuation buyer and a low-cost seller one-to-one. When their proportion is high, however, the platform finds it optimal to introduce distortion by matching some of the high-valuation buyers to high-cost sellers.[2] We further show that if (and only if) it entails BSM, the optimal mechanism is not *second-best* either in the sense it does not maximize social welfare *subject to* the agents' incentive constraints.

Having established the possible inefficiency of the optimal matching mechanism that allows for free strategic interactions between the matched agents, we proceed to the study of alternative scenarios in which the platform exercises stronger control over their interactions. The first such scenario involves *outcome-based pricing,* whereby subscription fees are contingent on whether or not a transaction has been consummated in a match. We show that this model restores PAM as the optimal matching rule, and eliminates the aforementioned efficiency distortion. On the other hand, the optimal mechanism extracts full informational rents from the subscribers when the market is symmetric. It follows that outcome-based pricing achieves first-best efficiency, but leads to a heavily skewed welfare distribution. We will discuss that also in this scenario, it is the strategic incentives of the agents that shape the optimal mechanism.

In our second alternative scenario, the platform exercises full control over both the allocation and price of the good as a function of the subscription plans chosen by the matched agents. The

[2]In the Appendix, we examine the effect of the game protocol by studying the optimal mechanism with seller-offer bargaining, and show that this conclusion for the symmetric market continues to hold.

platform in this model hence leaves no room for strategic behavior by either agent. In this scenario, we show that the platform employs PAM as the optimal matching rule. For some type distributions, however, the optimal mechanism introduces distortions by blocking efficient transactions between a low-cost seller and a low-valuation buyer, or a high-cost seller and a high-valuation buyer. This suggests that the match value can vary with the underlying populations even without strategic interactions.

Regulation authorities and policy makers are generally concerned about monopolistic platforms exercising favoritism to some of their users.[3] Our analysis identifies if and how a monopolistic platform creates social inefficiencies when it screens potential subscribers. In our context, favoritism is interpreted as a favor given to the agents who pay a premium for their subscription. More specifically, the favor takes the form of a higher probability of being matched with a more efficient type on the other side of the market. When free strategic interactions are allowed between the matched agents on a platform, we show that the optimal mechanism may create distortions through the choice of an inefficient matching rule. This suggests that favoritism itself can take an inefficient form. On the other hand, if no such strategic interactions are allowed, the optimal mechanism may create efficiency distortions by blocking efficient transactions. This suggests that favoritism by itself is efficient, but inefficiencies arise from the excessive power of the platform to block some efficient transactions. Our analysis hence suggests the intricate nature of the efficiency property of the optimal matching mechanism in relation to the degree of control over strategic behavior by the agents on a platform.

While our main analysis of a trading platform revolves around *two-sided competition* where agents from two sides of the market engage in strategic bargaining over trade surplus, another important form of strategic interactions on a platform is *one-sided,* where it is the agents from the same side of the market who compete against each other. A prominent example of one-sided competition is an auction platform, where buyers of a good competitively bid for a seller's good. By considering a stylized model of an auction platform in which up to two buyers are matched with a single seller, we show that a different type of assortative matching emerges as the key matching rule. Specifically, we show that the optimal matching rule between two buyers is negatively assortative in the sense that a high-valuation buyer is matched with a low-valuation buyer as much as possible. The intuition is simple and generalizes beyond the specific model we consider: A high-valuation buyer is willing to pay a premium for their subscription plan only if it reduces the chance of tough competition with another high-valuation buyer. We show that the optimal mechanism indeed takes advantage of such an incentive.

The paper is organized as follows. In Section 2, we discuss the related literature. Section 3 introduces a model of a trading platform. A characterization of an optimal matching mechanism is given in Section 4 under the buyer-offer bargaining protocol. Section 5 presents a model of outcome-based pricing, and Section 6 presents a model of a matching mechanism in which the platform has full control over the allocation and price of the good. We discuss one-sided competition in Section 7 with the formal presentation given in the Appendix. We conclude with a discussion in Section 8. The Appendix collects all the proofs.

[3]For example, the authorities suspect that Amazon gives better visibility and search rankings to sellers that pay to use its ads and fulfillment network for shipping products. ("How Each Big Tech Company May be Targeted by Regulators," Jack Nicas, Karen Weise, and Mike Isacc, *New York Times,* September 8, 2019.)

2 Related Literature

The key component of the matching models in the literature is a *production function,* which determines the value of a match to each member as a function of their types. The critical observation by Becker (1973) is that when the production function is supermodular on the set of agents' type profiles, the match allocation is in the core if it entails positive assortative matching (PAM), whereby the highest type on one side is matched with the highest type on the other side, the second-highest type is matched with the second-highest type, *etc.* Legros & Newman (2002) identify a condition on the production function weaker than supermodularity for PAM to be the core outcome, and Legros & Newman (2007) establish a sufficient condition for PAM to be in the core in an environment without full transferability of payoffs between the matched agents. Shimer & Smith (2000) show that the supermodularity of a production function is necessary but not sufficient for PAM to be the equilibrium outcome of a search model where each agent engages in continuous-time search for his partner.

The use of an exogenously specified production function for each agent is maintained in the literature on platforms that match agents with private types. Damiano & Li (2007) and Hoppe et al. (2011) study matching of agents with heterogeneous quality in two-sided markets when the match quality is the product of the qualities of its members (and hence supermodular), and Gomes & Pavan (2016) study efficient as well as profit-maximizing platforms for non-exclusive many-to-many matching in a two-sided market when the value of a match to an agent is the product of his value type and the average salience type of the matched agents. Board (2009) considers the problem of grouping agents with variable qualities, and identifies a profit maximizing group structure under various assumptions on the form of the production function.[4] In our model, aggregate surplus generated by trading within a match is a function of the types of the matched agents, but the division of surplus is determined endogenously in equilibrium. Specifically, the matched agents play a Bayesian equilibrium of a non-cooperative game, and the distribution of their types in each match is controlled by the matching rule of the platform.

Strategic interactions among subscribing agents are studied by Tamura (2016) and Birge et al. (2019) in models of monopolistic trading platforms. Tamura (2016) considers a platform that matches a single seller with multiple buyers when the seller is privately informed about the quality of his good, and the buyers' private types are affiliated with the quality. Tamura (2016) assumes that the platform offers a single subscription price to each side of the market, and shows that its subscription revenue is higher under the first-price auction than under the second-price auction. Birge et al. (2019) consider a platform that matches sellers and buyers one-to-one under the constraint that some type pairs are not feasible. When the agents' types are public, Birge et al. (2019) show that uniform pricing is suboptimal, and evaluate the optimal subscription revenue from discriminatory pricing under complete information. Unlike these models, our model features screening of privately informed agents through discriminatory pricing and matching. Furthermore, we present explicit characterizations of optimal matching mechanisms in the presence of strategic interactions among subscribers.

[4] Marx & Schummer (2019) focus on the stability of the matching rule when agents have heterogeneous preferences over other agents on the other side of the market, and study the optimal mechanism that offers a single price for each side.

It is possible to interpret our model as one of information design by a platform. In the information design literature, a principal controls the type distribution of each player so as to maximize his own payoff. In the Bayesian persuasion model of Kamenica & Gentzkow (2011), for example, a principal controls the distribution of signals about the state of the world that a decision maker observes, and attempts to maximize the probability with which the decision maker chooses the action preferred by the principal. In the multi-player information design model of Bergemann & Morris (2016), a principal likewise controls the distribution of signals about the state which are privately observed by the players. The principal in this case attempts to induce the (Bayes) correlated equilibrium of the game that is most preferable to him. In the present setting, the matching rule of the platform also controls the type distribution of the agents in the trading game, and is used to induce the Bayesian equilibrium that would maximize the subscription revenue. The key difference is that while the principal in models of information design generates information and reveals it to the agents, the platform in our model collects information from the agents: The matching rule and subscription fees are chosen so that they induce truth-telling in reporting of private signals by the agents.

3 Model of a Trading Platform

The market consists of two sides A and B as well as a monopolistic provider of a trading platform. The side A is a unit mass of sellers of an indivisible good, and the side B is a unit mass of buyers of the good. Each seller has a single unit of the good and each buyer has a unit demand for the good. An agent on either side has access to another agent on the other side only through subscription to the platform. Specifically, the platform sets fees for subscription, and then forms a one-to-one match between a subscribing seller and a subscribing buyer. A seller's cost of providing the good is denoted α, and a buyer's valuation is denoted β. The types α and β are private information of the agents, and are randomly drawn from the binary sets $A = \{\alpha_1, \alpha_2\}$ and $B = \{\beta_1, \beta_2\}$, respectively.[5] We suppose that the types are overlapped $\alpha_1 < \beta_1 < \alpha_2 < \beta_2$ so that no efficient transaction is feasible when a match involves a type β_1 buyer and a type α_2 seller (Figure 1). For simplicity, we assume that

$$\alpha_1 = 0,\ \alpha_2 = 1,\ \beta_1 = \gamma, \text{ and } \beta_2 = 1 + \gamma \text{ for } \gamma \in (0, 1).^{6}$$

A seller is type α_i with probability $\lambda_i \in (0, 1)$ and a buyer is type β_i with probability $\mu_i \in (0, 1)$ for $i = 1, 2$. The type realizations are independent across agents so that we may identify λ_i as the proportion of type α_i sellers on side A, and μ_i as the proportion of type β_i buyers on side B.

Once matched, a seller and buyer play a *trading game.* In the main body of analysis, we suppose that this trading game takes the form of a take-it-or-leave-it offer from the buyers. More generally, however, we can express it as a double-auction in which both players simultaneously submit bids. Let z_A and z_B denote a seller's and a buyer's bids, respectively, and $z = (z_A, z_B)$ be the bid

[5] Note that the symbols A and B are used to denote the sides of the market as well as the sets of types of agents on each side. A natural extension would involve a continuum of types on each side. Despite the clear difficulty with such an extension, we believe that it adds limited insight.

[6] The argument goes through with no qualitative change without this simplification.

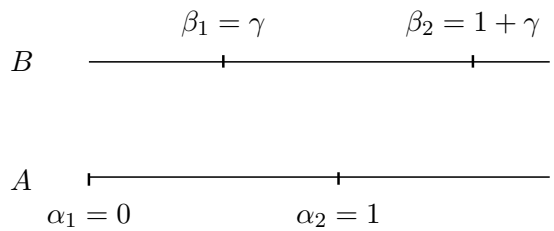

Figure 1: Costs and valuations

profile. The transaction is consummated if and only if $z_B \geq z_A$, and the transaction price equals $k(z) \equiv kz_A + (1-k)z_B$ for some constant $k \in [0,1]$. Accordingly, under the bid profile $z \in \mathbf{R}^2_+$, the payoff $g_A(z,\alpha)$ of a seller of type α, and the payoff $g_B(z,\beta)$ of a buyer of type β in the trading game are given by

$$g_A(z,\alpha) = \begin{cases} k(z) - \alpha & \text{if } z_A \leq z_B \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad g_B(z,\beta) = \begin{cases} \beta - k(z) & \text{if } z_A \leq z_B \\ 0 & \text{otherwise.} \end{cases}$$

In our main scenario of buyer-offer bargaining with sequential moves, we set $k = 0$ so that the transaction price always equals the buyer's bid: $k(z) = z_B$. In this case, the weakly dominant response by a seller of cost type α is to accept if and only if $z_B \geq \alpha$.[7]

Note that there is surplus from trade unless a high-cost seller (α_2) is matched with a low-valuation buyer (β_1). If we denote by

$$f(\alpha,\beta) = (\beta - \alpha)\,\mathbf{1}_{\{\beta > \alpha\}} \tag{1}$$

the aggregate surplus of trade when a type α seller is matched with a type β buyer, then f is *supermodular* since it satisfies

$$f(\alpha_1,\beta_2) + f(\alpha_2,\beta_1) = \beta_2 - \alpha_1 > (\beta_2 - \alpha_2) + (\beta_1 - \alpha_1) = f(\alpha_2,\beta_2) + f(\alpha_1,\beta_1).^{8} \tag{2}$$

Formally, the *matching mechanism* Γ of the platform consists of menus of *subscription plans* $\theta_A = (\theta^1_A, \theta^2_A)$ for side A and $\theta_B = (\theta^1_B, \theta^2_B)$ for side B. Each subscription plan is characterized by the price and the probabilities with which a subscriber of that plan is matched with subscribers of different plans on the other side. We denote by $\tau_A(\theta^i_A)$ the price of plan θ^i_A on side A, and by $\tau_B(\theta^j_B)$ the price of plan θ^j_B on side B. Let also $p_B(\theta^j_B \mid \theta^i_A)$ denote the probability with which a subscriber of plan θ^i_A is matched with a subscriber of plan θ^j_B, and $p_A(\theta^i_A \mid \theta^j_B)$ denote the probability with which a subscriber of plan θ^j_B is matched with a subscriber of plan θ^i_A. Clearly, these probabilities cannot be chosen arbitrarily and must satisfy some consistency requirements. For the presentation of these consistency requirements, we assume in what follows that plan θ^i_A is chosen by type α_i sellers, and plan θ^j_B is chosen by type β_j buyers. We can hence interpret plan θ^1_A for side A and plan θ^2_B for side B as *premium* subscription plans designed for more efficient types, whereas plan θ^2_A for

[7] In the Appendix, we analyze the trading game with seller-offer bargaining. This corresponds to setting $k = 1$ or $k(z) = z_A$.

[8] We consider the partial ordering over the set of type profiles (α,β) that is induced by $\alpha_2 \prec \alpha_1$ and $\beta_1 \prec \beta_2$.

side A and plan θ_B^1 for side B as *standard* subscription plans designed for less efficient types. The conditions which make these choices incentive compatible will be discussed after the description of the consistency conditions. Under our supposition, p_A and p_B must satisfy

$$\sum_{i=1}^{2} p_A(\alpha_i \mid \beta_j) = 1 \text{ for } j = 1,\, 2, \qquad \sum_{j=1}^{2} p_B(\beta_j \mid \alpha_i) = 1 \text{ for } i = 1,\, 2, \tag{3}$$

$$\lambda_i p_B(\beta_j \mid \alpha_i) = \mu_j\, p_A(\alpha_i \mid \beta_j) \text{ for } i,\, j = 1,\, 2, \tag{4}$$

$$\mu_1 p_A(\alpha_1 \mid \beta_1) + \mu_2 p_A(\alpha_1 \mid \beta_2) = \lambda_1, \tag{5}$$

$$\lambda_1 p_B(\beta_2 \mid \alpha_1) + \lambda_2 p_B(\beta_2 \mid \alpha_2) = \mu_2. \tag{6}$$

(3) simply states that each agent on one side is matched with either type on the other side.[9] (4) should hold since both sides represent the probability that type α_i and type β_j are matched with each other. (5) ensures that the total probability that either buyer type is matched with type α_1 sellers should equal the proportion of α_1 on side A, and likewise (6) ensures that the total probability that either seller type is matched with type β_2 buyers should equal the proportion of β_2 on side B. We refer to (5) and (6) as the *Bayes plausibility* conditions following Kamenica & Gentzkow (2011), who use the terminology for the corresponding condition in their analysis of Bayesian persuasion. We now introduce the probability distribution $p = (p_{ij})_{i,j=1,2}$ over the set $A \times B$ of type profiles defined by

$$p_{ij} = \lambda_i p_A(\beta_j \mid \alpha_i) = \mu_j p_B(\alpha_i \mid \beta_j).$$

By (3)-(6), p is well-defined and represents the proportion of (α_i, β_j) pairs in all matches. We refer to p as a *matching rule* p, and use it to express p_A and p_B in the subscription plans. Denote by P the set of feasible matching rules:

$$P \equiv \Big\{ p \in \Delta(A \times B) : \sum_{i=1}^{2} p_{ij} = \mu_j \text{ for } j = 1,\, 2, \text{ and } \sum_{j=1}^{2} p_{ij} = \lambda_i \text{ for } i = 1,\, 2 \Big\}.$$

The matching rule p defines the Bayesian game played by a pair of matched agents. Let $\sigma = (\sigma_A, \sigma_B)$ be a (pure) Bayes Nash equilibrium (BNE) of the incomplete information game that follows when the matched agents on both sides have chosen the plans designed for them. Since the joint distribution of type profiles equals p in this game, σ should satisfy

$$\pi_A(\sigma, \alpha_i) \equiv \sum_{j=1}^{2} p_B(\beta_j \mid \alpha_i)\, g_A(\sigma_A(\alpha_i), \sigma_B(\beta_j), \alpha) \geq \sum_{j=1}^{2} p_B(\beta_j \mid \alpha_i)\, g_A(z_A, \sigma_B(\beta_j), \alpha_i)$$
$$\text{for any } z_A \geq 0 \text{ and } i = 1,\, 2,$$

and

$$\pi_B(\sigma, \beta_j) \equiv \sum_{i=1}^{2} p_A(\alpha_i \mid \beta_j)\, g_B(\sigma_A(\alpha_i), \sigma_B(\beta_j), \beta_j) \geq \sum_{i=1}^{2} p_A(\alpha_i \mid \beta_j)\, g_B(\sigma_A(\alpha_i), z_B, \beta_j)$$
$$\text{for any } z_B \geq 0 \text{ and } j = 1,\, 2.$$

[9] We do not consider the possibility that some agents are left unmatched since it is suboptimal from the point of the platform as can be readily verified.

Throughout the main body of our analysis, we use σ to denote the BNE of the buyer-offer bargaining game in which the seller accepts the buyer's offer if and only if it is greater than or equal to his cost.[10]

Finally, we also rewrite the subscription price as a function of types $t_A(\alpha_i) = \tau_A(\theta_A^i)$ and $t_B(\beta_j) = \tau_B(\theta_B^j)$ given our assumption that type α_i chooses plan θ_A^i and that type β_j chooses plan θ_B^j (i, $j = 1, 2$). We can think of $t_A(\alpha)$ as the transfer from a seller who reports type α, and $t_B(\beta)$ as the transfer from a buyer who reports type β.

The matching mechanism Γ, which is now expressed in terms of a matching rule p and transfer rule t, is *incentive compatible* (IC) if no unilateral deviation in reporting and action choice is profitable:

$$\pi_A(\sigma, \alpha_i) - t_A(\alpha_i) \geq \sum_{j=1}^{2} p_B(\beta_j \mid \alpha_{i'})\, g_A(z_A, \sigma_B(\beta_j), \alpha_i) - t_A(\alpha_{i'})$$
$$\text{for every } i,\ i' = 1,\ 2 \text{ and } z_A \geq 0,$$

and

$$\pi_B(\sigma, \beta_j) - t_B(\beta_j) \geq \sum_{i=1}^{2} p_A(\alpha_i \mid \beta_{j'})\, g_B(\sigma_A(\alpha_i), z_B, \beta_j) - t_B(\beta_{j'})$$
$$\text{for every } j,\ j' = 1,\ 2 \text{ and } z_B \geq 0.$$

Γ is *individually rational* (IR) if truthful reporting yields at least as much as the reservation utility, which is normalized to zero:

$$\pi_A(\sigma, \alpha_i) - t_A(\alpha_i) \geq 0 \quad \text{for every } i, \text{ and}$$
$$\pi_B(\sigma, \beta_j) - t_B(\beta_j) \geq 0 \quad \text{for every } j.$$

The IC and IR conditions together guarantee the choices of the subscription plans as assumed.

The platform's subscription revenue $R(\Gamma)$ under the mechanism Γ equals the sum of revenue from both sides of the market:

$$R(\Gamma) = \underbrace{\sum_{i=1}^{2} \lambda_i t_A(\alpha_i)}_{\text{revenue from sellers}} + \underbrace{\sum_{j=1}^{2} \mu_j t_B(\beta_j)}_{\text{revenue from buyers}} .$$

A matching mechanism Γ is *optimal* if it maximizes revenue subject to the IC and IR conditions. When the mechanism Γ is IC and IR, its *efficiency* is defined by

$$W(\Gamma) = \sum_{i,j} (\beta_j - \alpha_i)\, p_{ij}\, \mathbf{1}_{\{\sigma_A(\alpha_i) \leq \sigma_B(\beta_j)\}}. \tag{7}$$

[10] Note that there exist other BNE's in the buyer-offer bargaining game in which the seller chooses a weakly dominated strategy. The optimal mechanism depends on which BNE is played in the trading game. In Appendix A.2, we present the analysis of an optimal mechanism when the agents play the BNE that is most preferred by the platform.

A *positive assortative matching* (PAM) rule p matches the low-cost sellers with the high-valuation buyers as much as possible: $p \in \operatorname{argmax}_{\hat{p} \in P} \hat{p}_A(\alpha_1 \mid \beta_2)$, or equivalently,

$$(p_{11}, p_{12}, p_{21}, p_{22}) = \begin{cases} (0, \lambda_1, \mu_1, \mu_2 - \lambda_1) & \text{if } \lambda_1 \leq \mu_2, \\ (\lambda_1 - \mu_2, \mu_2, \lambda_2, 0) & \text{if } \lambda_1 > \mu_2. \end{cases} \tag{8}$$

Note that even under PAM, some high-valuation buyers are matched with high-cost sellers (*i.e.,* $p_{22} > 0$) if $\lambda_1 < \mu_2$, and some low-cost sellers are matched with low-valuation buyers (*i.e.,* $p_{11} > 0$) if $\lambda_1 > \mu_2$. When the market is *symmetric* (*i.e.,* $\lambda_1 = \mu_2$), on the other hand, PAM creates no mismatch (*i.e.,* $p_{11} = p_{22} = 0$). In the literature, PAM is associated with both social efficiency and optimality from the platform's perspective. In our model, it is also intuitive and readily verified that PAM maximizes the aggregate surplus from transactions as formally stated in the following proposition.

Proposition 3.1 *If the matching rule p maximizes $\sum_{i,j} p_{ij}(\beta_j - \alpha_i)\, \mathbf{1}_{\{\beta_j > \alpha_i\}}$, then it is PAM. Furthermore, the maximal social surplus equals*

$$W^* = \begin{cases} \gamma\mu_2 + \lambda_1 & \textit{if } \lambda_1 \leq \mu_2, \\ \gamma\lambda_1 + \mu_2 & \textit{if } \lambda_1 > \mu_2. \end{cases} \tag{9}$$

Proposition 3.1 shows that if a matching mechanism Γ entails PAM, it achieves social optimum provided that all efficient transactions take place between matched agents. The only question in that case would be how it distributes welfare between the agents and the platform. On the other hand, if Γ entails matching rule other than PAM, or if Γ entails PAM but blocks efficient transactions, it implies efficiency distortions.

Before proceeding, we mimic the production function approach in the literature and identify the optimal mechanism in that formulation. Suppose specifically that the match values for the sellers and buyers are respectively given by

$$f_A(\alpha_i, \beta_j) = \begin{cases} (1-k)(\beta_j - \alpha_i) & \text{if } \beta_j > \alpha_i, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad f_B(\alpha_i, \beta_j) = \begin{cases} k(\beta_j - \alpha_i) & \text{if } \beta_j > \alpha_i, \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

Since $f_A = (1-k)\, f$ and $f_B = kf$ for the aggregate surplus f defined in (1), both f_A and f_B are supermodular by (2). The following proposition replicates the finding in the literature and shows that the optimal mechanism entails PAM, and achieves first-best efficiency.

Proposition 3.2 *(Optimal matching with production functions) Suppose that the agents' match values are given by (10). If Γ is an optimal mechanism, then p is PAM, and the associated revenue is given by*

$$R(\Gamma) = \begin{cases} \frac{1-k}{\lambda_2}\left\{\gamma(\mu_2 - \lambda_1) + \lambda_1\mu_1\right\} + k\lambda_1 & \textit{if } \lambda_1 \leq \mu_2, \\ \frac{k}{\mu_1}\left\{\gamma(\lambda_1 - \mu_2) + \lambda_2\mu_2\right\} + (1-k)\mu_2 & \textit{if } \lambda_1 > \mu_2. \end{cases} \tag{11}$$

In particular, when the market is symmetric ($\lambda_1 = \mu_2$), $R(\Gamma) = \lambda_1 = \mu_2$.

4 Optimal Mechanism under Strategic Interaction

In the game where a buyer makes a take-it-or-leave-it offer (*i.e.,* $k(z) = z_B$ for every z), let $\sigma = (\sigma_A, \sigma_B)$ be the BNE in which the seller of cost type α plays the weakly dominant strategy of accepting the offer z_B if and only if $z_B \geq \alpha$: $\sigma_A(\alpha_i) = \alpha_i$ for every $i = 1, 2$. σ_B is a best response against σ_A specified as follows. Let $z_B^*(\beta, \tilde{p}_A)$ be the seller's optimal bid against σ_A when his true type is β, and his belief about the type of the matched seller is given by $\tilde{p}_A$:

$$z_B^*(\beta, \tilde{p}_A) \in \operatorname{argmax}_{z_B \in \mathbf{R}_+} \sum_{\alpha \in A} \tilde{p}_A(\alpha)\, g_B((\sigma_A(\alpha), z_B), \beta).$$

Explicitly, $z_B^*(\beta, \tilde{p}_A)$ can be written as

$$z_B^*(\beta, \tilde{p}_A) = \begin{cases} \alpha_1 & \text{if } \beta = \beta_1, \text{ or if } \beta = \beta_2 \text{ and } \tilde{p}_A(\alpha_1) \geq \frac{\beta_2 - \alpha_2}{\beta_2 - \alpha_1} = \frac{\gamma}{1+\gamma}, \\ \alpha_2 & \text{if } \beta = \beta_2 \text{ and } \tilde{p}_A(\alpha_1) < \frac{\beta_2 - \alpha_2}{\beta_2 - \alpha_1} = \frac{\gamma}{1+\gamma}. \end{cases}$$

In other words, the only viable bid for the low-valuation buyer β_1 is α_1, whereas the optimal bid for the high valuation buyer β_2 varies with his belief about the type of the matched seller: He either bids α_1 and has the low-cost seller accept it, or bids α_2 and has both seller types accept it. After truthful reporting, his belief is given by $\tilde{p}_A = p_A(\cdot \mid \beta)$ so that the BNE strategy σ_B can be defined by

$$\sigma_B(\beta) = z_B^*(\beta, p_A(\cdot \mid \beta)) \quad \text{for every } \beta.$$

The agents play the BNE $\sigma = (\sigma_A, \sigma_B)$ of the trading game after truthful reporting.

Characterization of an optimal mechanism requires the introduction of two more matching rules other than PAM. First, a matching rule p is *random* (RM) if it matches agent types according to their proportions in the populations:

$$(p_{11}, p_{12}, p_{21}, p_{22}) = (\lambda_1\mu_1, \lambda_1\mu_2, \lambda_2\mu_1, \lambda_2\mu_2).$$

A matching rule p is *B-squeeze* (BSM) if the distribution of seller types faced by a low-valuation buyer β_1 equals the threshold value $\frac{\gamma}{1+\gamma}$ for the high-valuation buyer. In other words, when a high-valuation buyer β_2 misreports his type as β_1 in the subscription stage, then in the trading game, his expected surplus γ from the high offer (α_2) is exactly the same as his expected surplus $p_A(\alpha_1 \mid \beta_1)(1+\gamma)$ from the low offer (α_1). Formally, when $(1+\gamma)\lambda_1 + \mu_1 \leq 1 + \gamma$, a matching rule p is BSM if

$$(p_{11}, p_{12}, p_{21}, p_{22}) = \Big(\frac{\gamma}{1+\gamma}\,\mu_1,\ \lambda_1 - \frac{\gamma}{1+\gamma}\,\mu_1,\ \frac{1}{1+\gamma}\,\mu_1,\ \lambda_2 - \frac{1}{1+\gamma}\,\mu_1\Big).$$

We can verify that if (and only if) $(1+\gamma)\lambda_1 + \mu_1 \leq 1 + \gamma$ and $\lambda_1 \geq \frac{\gamma}{1+\gamma}$, the probability that a type β_2 buyer is matched with a type α_1 seller is higher than the probability that a type β_1 buyer is matched with α_1. Unlike PAM, however, BSM does not maximize the probability p_{12} that α_1 is matched with β_2.

The characterization of the optimal mechanism is given by the following proposition and is illustrated in Figure 2.

Proposition 4.1 *Suppose that Γ is an optimal matching mechanism with buyer-offer bargaining. Then its matching rule p and revenue $R(\Gamma)$ are given as follows.*

a. If $(1+\gamma)\lambda_1 + \mu_1 > 1+\gamma$*, then* p *is PAM and* $R(\Gamma) = (1+\gamma)\lambda_1 - \frac{\lambda_1 - \mu_2}{\mu_1}$.

b. If $\mu_1 > \frac{\gamma}{1+\gamma}$, $\lambda_1 + \gamma\mu_1 > \gamma$*, and* $(1+\gamma)\lambda_1 + \mu_1 \leq 1+\gamma$*, then* p *is PAM and*

$$R(\Gamma) = \begin{cases} (1+\gamma)\lambda_1 - \mu_2\gamma & \text{if } \lambda_1 \leq \mu_2, \\ \frac{\lambda_1 - \mu_2}{\mu_1}\gamma + \mu_2 & \text{if } \lambda_1 > \mu_2, \end{cases}$$

c. If $\lambda_1 > \frac{\gamma}{1+\gamma}$, $(1+\gamma)\lambda_1 + \mu_1 \leq 1+\gamma$*, and* $\mu_1 \leq \frac{\gamma}{1+\gamma}$*, then* p *is BSM and* $R(\Gamma) = \lambda_1(1+\gamma) - \frac{\gamma}{1+\gamma}$.

d. If $\lambda_1 \leq \frac{\gamma}{1+\gamma}$ *and* $\lambda_1 + \gamma\mu_1 \leq \gamma$*, then* p *is RM and* $R(\Gamma) = \gamma\lambda_1$.

Proposition 4.1 shows the existence of markets in which PAM is suboptimal and hence optimality diverges from social efficiency. This presents a sharp contrast with the prevailing intuition in the literature on the optimality of PAM.

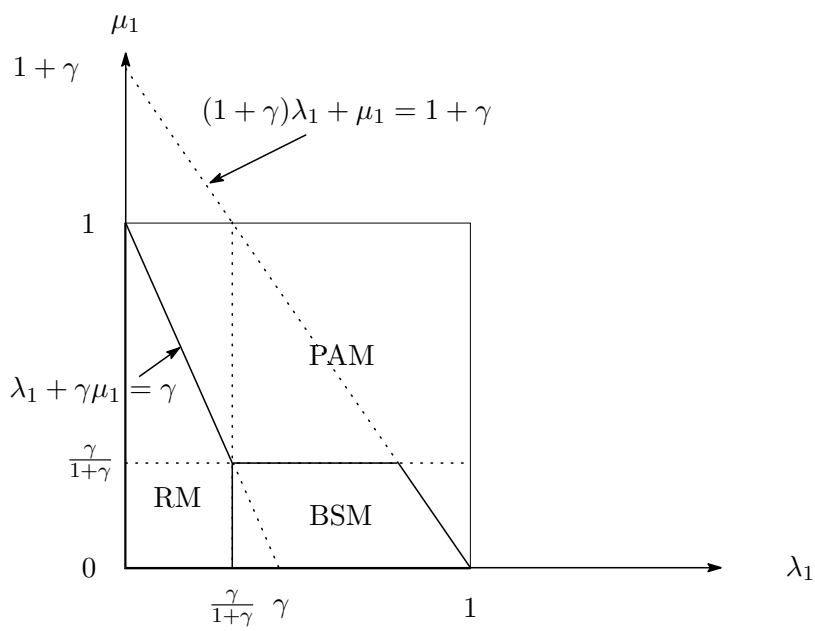

Figure 2: Optimal Matching with Buyer-Offer Bargaining

As mentioned in the Introduction, the suboptimality of PAM is a direct consequence of the strategic interactions between the subscribing agents. To see this, consider first the optimality of BSM. As is standard in the screening models, the IR condition for the type β_1 buyers and the IC condition for the type β_2 buyers bind. Hence, the transfer $t_B(\beta_1)$ for β_1 equals his expected surplus from trade: $t_B(\beta_1) = p_A(\alpha_1 \mid \beta_1)(\beta_1 - \alpha_1) = p_A(\alpha_1 \mid \beta_1)\gamma$. On the other hand, the surplus from trade for type β_2 after misreporting is given by $p_A(\alpha_1 \mid \beta_1)(\beta_2 - \alpha_1) = p_A(\alpha_1 \mid \beta_1)(1+\gamma)$ if $p_A(\alpha_1 \mid \beta_1) \geq \frac{\gamma}{1+\gamma}$ (from the offer α_1 accepted only by α_1), and $\beta_2 - \alpha_2 = \gamma$ if $p_A(\alpha_1 \mid \beta_1) \leq \frac{\gamma}{1+\gamma}$ (from the offer α_2 accepted by both seller types). Since β_2's IC condition is binding, his payoff from subscription with truth-telling equals the payoff he would obtain by misreporting:

$$\pi_B(\sigma, \beta_2) - t_B(\beta_2) = \begin{cases} \gamma - t_B(\beta_1) = (1 - p_A(\alpha_1 \mid \beta_1))\gamma & \text{if } p_A(\alpha_1 \mid \beta_1) \leq \frac{\gamma}{1+\gamma}, \\ p_A(\alpha_1 \mid \beta_1)(1+\gamma) - t_B(\beta_1) = p_A(\alpha_1 \mid \beta_1) & \text{if } p_A(\alpha_1 \mid \beta_1) \geq \frac{\gamma}{1+\gamma}. \end{cases}$$

As indicated in the right panel of Figure 3, this surplus is minimized when $p_A(\alpha_1 \mid \beta_1) = \frac{\gamma}{1+\gamma}$. When the proportion μ_2 of type β_2 is sufficiently high in the population (*i.e.*, $\mu_2 \geq \frac{1}{1+\gamma} \Leftrightarrow \mu_1 \leq \frac{\gamma}{1+\gamma}$ is low), hence, the platform finds it optimal to squeeze their informational rents by setting $p_A(\alpha_1 \mid \beta_1) = \frac{\gamma}{1+\gamma}$ (and then choosing $p_A(\alpha_1 \mid \beta_2)$ to satisfy Bayes plausibility).[11]

Consider next the optimality of RM. When the proportion λ_1 of low-cost sellers (α_1) in the population is low, the probability that type β_2 is matched with α_1 cannot be high, and hence type β_2 should optimally bid α_2 regardless of whether he reports his type truthfully or not. With IC for β_2 binding, however, we have $\pi_B(\sigma, \beta_2) - t_B(\beta_2) = \gamma - t_B(\beta_2) = \gamma - t_B(\beta_1)$ so that the buyer transfer must be independent of the report: $t_B(\beta_1) = t_B(\beta_2)$. This further implies that the matching rule must also be independent of the reported type, and hence that only RM is feasible.

The situation is different for the production function approach in Proposition 3.2. As can be readily seen, the surplus of type β_2 in this case is an increasing linear function of $p_A(\alpha_1 \mid \beta_1)$ (the left panel of Figure 3). It then follows that β_2's surplus is minimized when $p_A(\alpha_1 \mid \beta_1)$ is minimized. The optimality of PAM hence follows.

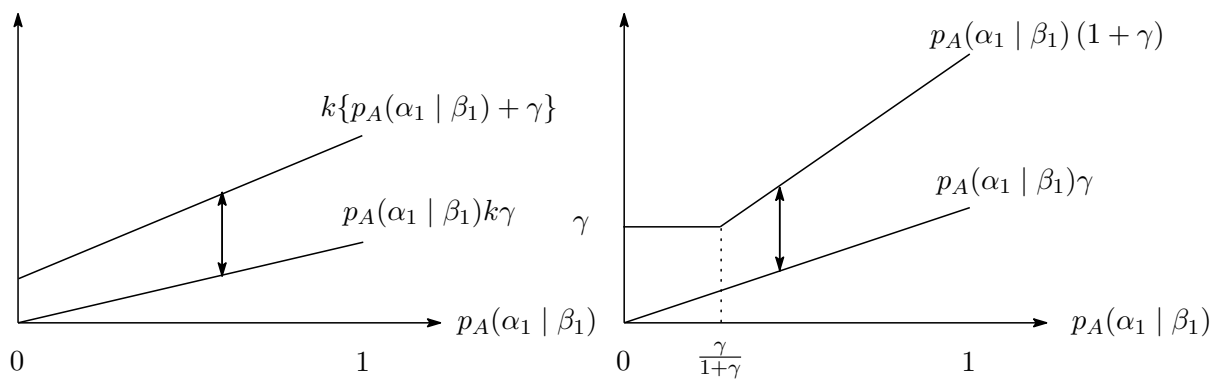

Figure 3: Informational rent of a type β_2 buyer in the non-strategic (left) and strategic (right) interaction

The following corollary to Proposition 4.1 identifies the optimal matching rule in a *symmetric* market in which the proportion of low-cost sellers equals that of high-valuation buyers: $\lambda_1 = \mu_2$.

Corollary 4.1 *Suppose that the market is symmetric with $\lambda_1 = \mu_2 = d$. Then the optimal mechanism Γ with buyer-offer bargaining entails PAM and yields d if $d \leq \frac{1}{1+\gamma}$, and BSM and yields $d(1+\gamma) - \frac{\gamma}{1+\gamma}$ if $d > \frac{1}{1+\gamma}$.*

Given that PAM in a symmetric market can match low-cost sellers and high-valuation buyers one-to-one, it is striking to observe that it is dominated by BSM when the proportion of those

[11] We can think of $t_B(\beta_1)$ as the base price and $t_B(\beta_2) - t_B(\beta_1)$ as a markup required for type β_2. Use of BSM corresponds to keeping the base price relatively high and the markup relatively low (so as to minimize the informational rents of β_2). On the other hand, when $\mu_1 > \frac{\gamma}{1+\gamma}$, BSM is dominated by PAM, which corresponds to minimizing the base price and maximizing the markup (while leaving higher informational rents to β_2). Intuitively, when μ_1 is high, keeping the base price $t_B(\beta_1) = p_A(\alpha_1 \mid \beta_1)\gamma$ above its minimum level is too costly given the Bayes plausibility condition $\mu_1 p_A(\alpha_1 \mid \beta_1) + \mu_2 p_A(\alpha_1 \mid \beta_2) = \lambda_1$.

types is high. In such a market, efficiency distortion is caused by optimal creation of mismatches by the platform.[12]

Given that the optimal mechanism is not always first-best efficient, is it *second-best efficient* in the sense that it maximizes the surplus from trade between the sellers and buyers subject to the agents' incentive constraints? In order to answer this question, we consider the efficient mechanism in the class of IC and IR mechanisms. Specifically, the *second-best* mechanism Γ maximizes $\max W(\Gamma)$ subject to the IC an IR constraints when the agents play the BNE σ specified above in the buyer-offer bargaining game.

Proposition 4.2 *Suppose that the matching mechanism Γ maximizes social welfare in the class of IC and IR mechanisms with buyer-offer bargaining. Then the associated matching rule p and the corresponding social welfare are given by*

a. If $\lambda_1 + \gamma\mu_1 > \gamma$ or $\lambda_1 \geq \frac{\gamma}{1+\gamma}$, then p is PAM and $W(\Gamma) = \begin{cases} (1+\gamma)\,\lambda_1 & \text{if } \lambda_1 \leq \mu_2, \\ \gamma\lambda_1 + \mu_2 & \text{if } \lambda_1 > \mu_2. \end{cases}$

b. If $\lambda_1 + \gamma\mu_1 \leq \gamma$ and $\lambda_1 \leq \frac{\gamma}{1+\gamma}$, then p is RM and $W(\Gamma) = \lambda_1(\gamma + \mu_2)$.

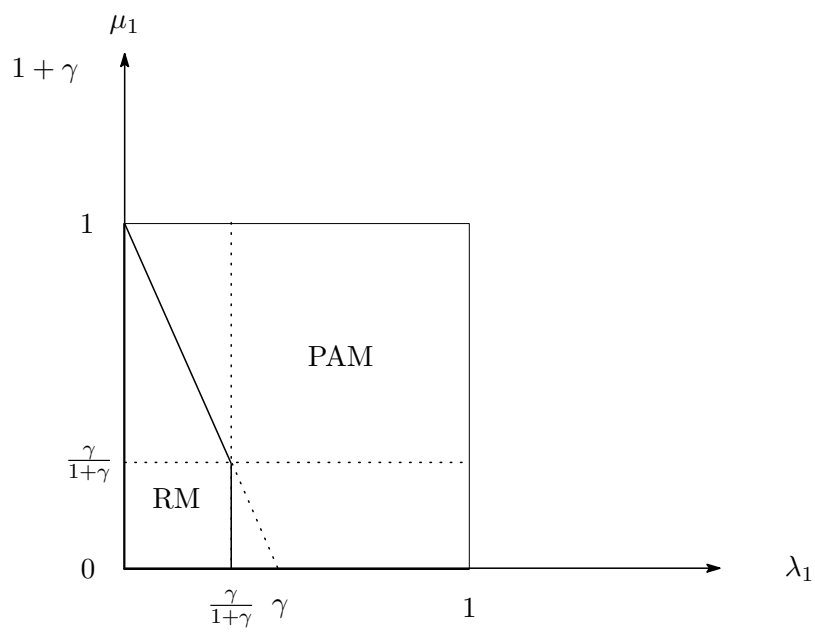

Figure 4: Second-best matching with buyer-offer bargaining

The second-best matching mechanism is illustrated in Figure 4. It can be seen from Propositions 3.1 and 4.2 that the second-best mechanism with buyer-offer bargaining is first-best efficient when

[12]Combining Proposition 3.2 and Corollary 4.1 reveals another interesting fact. Recall from Proposition 3.2 that for any value of $d = \lambda_1 = \mu_2$, the platform's optimal revenue under the production function approach equals d (as under PAM in Corollary 4.1). Hence, the maximal revenue under BSM with strategic interactions is higher than the maximal revenue with PAM under the production functions when $d > \frac{1}{1+\gamma}$.

$\lambda_1 \geq \mu_2$. Comparing Figures 2 and 4, we also see that revenue maximization is equivalent to welfare maximization except when the optimal mechanism entails BSM. Put differently, when the market has a high proportion of high-valuation buyers and a medium to large proportion of low-cost sellers, the subscription revenue is maximized at the expense of social welfare.

5 Outcome-Based Pricing

We have seen that free strategic interactions between the matched agents on a platform lead to inefficiencies of the optimal matching rule in some markets. How would this conclusion change when a platform exercises stronger control over the agents' interactions? In this section, we specifically examine a model in which the subscription fees are made contingent on a successful transaction. While such a fee scheme is frequently observed in reality, it requires a different institutional setting from our baseline model. First, the platform must be able to monitor the success of each transaction, and prevent secret exchange arrangements outside the system.[13] Second, the platform needs to enforce the payment of the fee even after the transaction. With the requirement of interim individual rationality as assumed elsewhere in the paper, it is clear that the optimal mechanism with outcome-contingent subscription fees generates a weakly higher revenue than the optimal mechanism in the baseline model since information about the transaction can always be ignored. The following proposition shows that the optimal outcome-contingent mechanism with buyer-offer bargaining entails PAM, and strictly dominates the optimal mechanism Γ of Proposition 4.1 in the baseline model when Γ entails non-PAM matching rules.[14] Denote by $t_A(\alpha)$ and $t_B(\beta)$ the transfer payments required of a type α seller and a type β buyer, respectively, when there is a successful transaction.[15]

Proposition 5.1 *Suppose that the transfer can be contingent on the outcome of the transaction. If $\tilde{\Gamma}$ is an optimal mechanism with buyer-offer bargaining that entails no transfer from sellers (*i.e., $t_A(\alpha_1) = t_A(\alpha_2) = 0$*), then the associated matching rule is PAM, and the revenue is equal to*

$$R(\tilde{\Gamma}) = \begin{cases} \lambda_1(1+\gamma) & \text{if } \lambda_1 \leq \mu_2, \\ \lambda_1(1+\gamma) - \frac{\lambda_1 - \mu_2}{\mu_1} & \text{if } \lambda_1 > \mu_2. \end{cases} \tag{12}$$

Furthermore, $R(\tilde{\Gamma}) = R(\Gamma)$ if $(1+\gamma)\lambda_1 + \mu_1 \geq 1+\gamma$, and $R(\tilde{\Gamma}) > R(\Gamma)$ otherwise.

Comparison of Propositions 3.1 and 5.1 shows that $\tilde{\Gamma}$ extracts the full social surplus when the market is symmetric (*i.e.,* $\lambda_1 = \mu_2$). The resurgence of PAM as the optimal matching rule in Proposition 5.1 is a consequence of changes in the agents' strategic incentives induced by the outcome-contingent fees. Specifically, for a high-valuation buyer (β_2), the surplus expected from a high bid α_2 and an aggressive bid α_1 depends on the transfer payment required in the event of

[13] For example, information about the agents' addresses need to be withheld so that physical trading of the good will not be possible until after the payment is made.

[14] Note that the optimal rule in the baseline model is non-PAM if $(1+\gamma)\lambda_1 + \mu_1 < 1+\gamma$.

[15] Positive subscription fees only in the event of a successful transaction ensure that the mechanism satisfies the stronger requirement of ex post individual rationality. The proposition also assumes that no subscription fee is required from a seller for simplicity. Consideration of a subscription fee for a seller with buyer-offer bargaining complicates the analysis substantially with little added insight.

a successful transaction. The platform finds it optimal to induce type β_2 to bid α_1 whether he reports his type truthfully or not: When $\lambda_1 \leq \mu_2$, for example, the optimal fee for type β_2 equals $t_B(\beta_2) = \beta_2 - \alpha_1 = 1+\gamma$, inducing type β_2 to bid α_1 after truthful reporting. Misreporting by β_2 is prevented by never matching β_1 with α_1 (*i.e.*, $p_A(\alpha_1 \mid \beta_1) = 0$), and setting $t_B(\beta_1) = \beta_2 - \alpha_2 = \gamma$. Minimization of $p_A(\alpha_1 \mid \beta_1)$ is equivalent to the use of PAM. The outcome-based pricing hence eliminates efficiency distortion, but induces a heavily unbalanced distribution of welfare.

6 Full Platform Control

As we have seen that partial control of the strategic interactions between the agents restores efficient PAM as the optimal matching rule, we now proceed to our second alternative scenario in which a platform exercises an even stronger control over their interactions. Specifically, we suppose that the platform exercises full control over the agents' interactions and dictates the terms of trade by specifying the allocation and price of the good as a function of their subscription plans. In other words, the platform leaves no room for the agents to take strategic behavior against their partners. While this may seem equivalent to the production function approach (Proposition 3.2), the match values in this case are endogenously determined in the process of optimization by the platform. We show that this yields a critical difference.

Formally, a matching mechanism Γ with full platform control is represented by a triplet (p, t, δ), where p and $t = (t_A, t_B)$ are the matching and transfer rules, respectively, as previously defined, whereas $\delta : A \times B \to [0, 1]$ is the *allocation rule* which determines the allocation of the good within a match: $\delta_{ij} \equiv \delta(\alpha_i, \beta_j)$ is the probability that a transaction takes place when a type α_i seller is matched with a type β_j buyer. The transfer rule t is inclusive of the price of the good.

Note that the mechanism Γ in this setting is efficient if and only if it entails PAM, and implements a transaction whenever it involves a low-cost seller or a high-valuation buyer. The efficient allocation rule $\delta = \bar{\delta}$ is explicitly given by

$$\bar{\delta} = (\bar{\delta}_{11}, \bar{\delta}_{12}, \bar{\delta}_{21}, \bar{\delta}_{22}) = \begin{cases} (*, 1, 0, 1) & \text{if } \lambda_1 \leq \mu_2, \\ (1, 1, 0, *) & \text{if } \lambda_1 > \mu_2, \end{cases} \tag{13}$$

where $*$ is any number between 0 and 1.[16] It can be readily verified that if the agents' types were observable, then the platform would use PAM and achieve first-best efficiency given in Proposition 3.1 while extracting full surplus from the agents. The following proposition shows that under incomplete information about the types, the optimal mechanism is not necessarily efficient.

Proposition 6.1 *Suppose that Γ is an optimal matching mechanism with full platform control. Then Γ entails PAM, and its revenue equals*

$$R(\Gamma) = \begin{cases} -\frac{\mu_2 - \lambda_1}{\lambda_2} + (1+\gamma)\mu_2 & \textit{if } \lambda_1 \leq \mu_2 \textit{ and } \lambda_1 \leq \frac{\gamma}{1+\gamma}, \\ (1+\gamma)\lambda_1 & \textit{if } \lambda_1 \leq \mu_2 \textit{ and } \lambda_1 > \frac{\gamma}{1+\gamma}, \\ -\frac{\lambda_1 - \mu_2}{\mu_1} + (1+\gamma)\lambda_1 & \textit{if } \lambda_1 > \mu_2 \textit{ and } \mu_1 \geq \frac{1}{1+\gamma}, \\ (1+\gamma)\mu_2 & \textit{if } \lambda_1 > \mu_2 \textit{ and } \mu_1 < \frac{1}{1+\gamma}. \end{cases}$$

[16] $\bar{\delta}_{11}$ is indeterminate since $p_{11} = 0$ under PAM when $\lambda_1 \leq \mu_2$, and $\bar{\delta}_{22}$ is indeterminate since $p_{22} = 0$ under PAM when $\lambda_1 > \mu_2$.

Furthermore, Γ *entails inefficient allocation rule* $\delta \neq \bar{\delta}$ *if* $\lambda_1 > \frac{\gamma}{1+\gamma}$ *and* $\mu_1 < \frac{1}{1+\gamma}$.

While the optimal matching mechanism in this setting matches the agent types efficiently, it deviates from (13) for some type distributions by blocking efficient transactions involving a low-cost seller and a low-valuation buyer (*i.e.,* $\delta_{11} = 0$), or a high-cost seller and a high-valuation buyer (*i.e.,* $\delta_{22} = 0$). Figure 5 illustrates the allocation rule $\delta = (\delta_{11}, \delta_{12}, \delta_{21}, \delta_{22})$ under Γ. We can think of the "no-transaction" specification by the optimal mechanism as a downward distortion imposed on the less efficient type to satisfy the IC condition of the efficient type more effectively. Because of the two-sided nature of the matching, this distortion has a consequence on the efficient type on the other side: Suppose for concreteness that $\lambda_1 < \mu_2$ so that there are more high-valuation buyers than low-cost sellers. Since then the less efficient type on side A (*i.e.,* type α_2) needs to be matched with the more efficient type on side B (*i.e.,* type β_2) even under PAM, type β_2 buyers must also face the sanction, implying $\delta_{22} = 0$ when $\lambda_1 < \mu_2$.

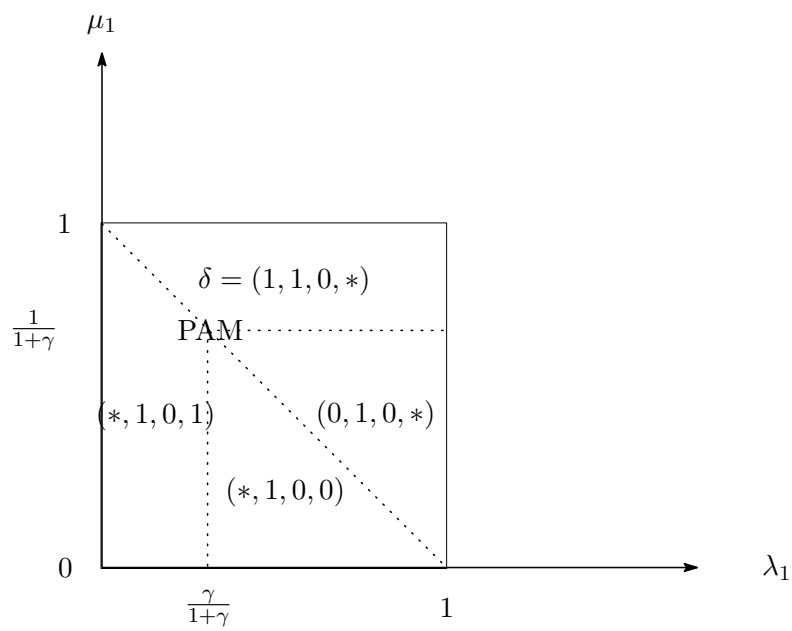

Figure 5: Optimal Matching Mechanism with Full Platform Control

PAM is optimal for every type distribution.

7 NAM under One-sided Competition

Many platforms in reality match a single agent on one side of the market with multiple agents on the other side. Unlike in our baseline model, this gives rise to another important form of strategic interaction where agents from the same side compete against each other. We refer to strategic interactions on a platform of agents from one side of the market as *one-sided competition,* and study the optimal matching mechanism in such an environment. Specifically, we formulate a model of an auction platform which matches each seller with up to two buyers. This model departs from the literature that discusses the negative externalities arising from subscription by more agents

from one side of the market.[17] In this model, which is presented formally in Appendix A.3, each seller has two types which now represent the quality of the good they possess. Each buyer, which is one of two valuation types as in the baseline model, competitively bids for the good of the matched seller upon observing its quality. We consider both the first-price and second-price auction as sales formats, and derive the optimal mechanism. In the case of the first-price auction, a high-valuation bidder randomizes his bid in BNE, and the distribution of his bids is higher in the sense of stochastic dominance when he believes that the other buyer is also a high-valuation type. We show that the optimal matching rule entails *negative assortative matching* (NAM) between the two buyers in the sense that a high-valuation buyer is matched with a low-valuation buyer as much as possible, and vice versa. The intuition is simple: A high-valuation buyer pays a premium for their subscription plan expecting that it will lead to a higher surplus. This is possible only if it reduces the chance of tough competition against another high-type buyer in the bidding stage. In order to maximize subscription revenue from the high-type buyer, hence, the platform finds it optimal to create as many high-low buyer pairs as possible. Once these buyer pairs are formed, the optimal mechanism then entails PAM between sellers and buyer pairs so that a high-quality seller is matched with a high-high buyer pair as much as possible, and then matched with a high-low buyer pair as much as possible. We expect that the simple intuition regarding NAM between competitors from the same side generalizes beyond our specific setting, and is characteristic of optimal matching under one-sided competition. Unlike in our baseline trading model, we show that this optimal matching rule is also first-best efficient: While NAM between buyers is again a consequence of the platform's rent extraction from high-valuation buyers, it does not result in allocative distortion since having them win as much as possible is in line with social efficiency.

8 Conclusion

Our analysis is motivated by the observation that few platforms in the real world dictate the terms of trade between their subscribers. When the subscribers play a strategic game against each other, we note that the value of a match created by the platform is endogenously determined by the BNE payoff of the game. Since the BNE depends on the type distribution of the players, there is room for the platform to manipulate the subscribers' beliefs through its matching rule and induce its preferred BNE. Exploration of this possibility provides new insights into the possible distortion created by a monopolistic platform. In a model of a one-to-one trading platform, we show that the optimal mechanism entails matching rule different from positive assortative matching (PAM) for some type distribution in the population, and discuss that the optimality of these non-PAM rules is a direct consequence of strategic interactions and is a source of efficiency distortion that has not been discussed in the literature. Specifically, the BSM matching rule results from the platform's attempt of to extract informational rents from high-valuation buyers under the strategic interaction. With outcome-based pricing where the subscription fees can depend not only on the subscription plans but also on the success of a transaction, we show that the optimal mechanism restores efficiency, but that the welfare distribution is heavily skewed with the platform extracting large rents from the agents. When the platform fully controls the allocation and price of the good,

[17] For example, more subscriptions by agents on one side lowers the probability that they get matched with the other side. See Belleflamme & Peitz (2019).

on the other hand, we shows that the optimal mechanism has PAM, but blocks efficient transactions in some markets. In a model of one-sided competition that matches each seller with two buyers, on the other hand, we show that the optimal matching involves negative assortative matching (NAM) between buyers. Although this again results from the attempt at rent extraction by the platform, we show that this is indeed consistent with first-best efficiency. The analysis of these models identifies the presence and source of inefficiencies involved in the optimal matching mechanism employed by a monopolistic two-sided platform.

We have assumed that the game between the two matched agents is one in which a buyer makes a take-it-or-leave-it offer and a seller accepts or rejects it. It is possible to think that the strategic interaction between the matched agents takes a different form. In Appendix A.1, we consider one such possibility where a seller instead makes a take-it-or-leave-it offer. We show that the optimal mechanism is the mirror image of that in our main model. Most importantly, we find that the inefficiency of the optimal mechanism persists in a symmetric market regardless of which side makes an offer.

As noted earlier, there exist BNE in the trading game other than the one considered in our main analysis. What would happen when the agents play the BNE that is most preferred by the platform? In Appendix A.2, we show that PAM is restored as the optimal matching rule, and hence the optimal mechanism is first-best efficient. We note however that such an equilibrium is not necessarily focal and weakly dominated if the game is buyer-offer or seller-offer bargaining.

One important question of empirical relevance concerns whether or not potential subscribers correctly anticipate the matching rule adopted by a platform. While correct anticipation of a mechanism is a standard assumption in equilibrium analysis, it is interesting to observe that potential subscribers often engage in extensive investigation into the relationship between their reported type and their expected match. For example, potential subscribers to a dating platform extensively search for the experiences of past subscribers through various review sites, and obtain very precise ideas about what to expect from the subscription.

There are a number of possible extensions of the model studied in this paper. Most importantly, while our analysis focuses on a monopoly platform, we do not discuss how it has acquired the proprietary status in the market. Monopolization can be the outcome of competition, and formal analysis is required on if and how competition among multiple platforms leads to monopolization. An important consideration in modeling such competition is the fact that the externalities between agents are also determined endogenously by their equilibrium decisions.

References

Becker, G. S. (1973), 'A theory of marriage: Part i', *Journal of Political Economy* **81**(4), 813–846.

Belleflamme, P. & Peitz, M. (2019), 'Managing competition on a two-sided platform', *Journal of Economics & Management Strategy* **28**(1), 5–22.

Bergemann, D. & Morris, S. (2016), 'Bayes correlated equilibrium and the comparison of information structures in games', *Theoretical Economics* **11**(2), 487–522.

Birge, J. R., Candogan, O., Chen, H. & Saban, D. (2019), Optimal commissions and subscriptions in networked markets, Research Papers 3742, Stanford University, Graduate School of Business.

Board, S. (2009), 'Monopolistic group design with peer effects', *Theoretical Economics* **4**(1), 89–125.

Damiano, E. & Li, H. (2007), 'Price discrimination and efficient matching', *Economic Theory* **30**(2), 243–263.

Gomes, R. & Pavan, A. (2016), 'Many-to-many matching and price discrimination', *Theoretical Economics* **11**(3), 1005–1052.

Hoppe, H. C., Moldovanu, B. & Ozdenoren, E. (2011), 'Coarse matching with incomplete information', *Economic Theory* **47**(1), 75–104.

Kamenica, E. & Gentzkow, M. (2011), 'Bayesian persuasion', *American Economic Review* **101**(6), 2590–2615.

Legros, P. & Newman, A. F. (2002), 'Monotone matching in perfect and imperfect worlds', *The Review of Economic Studies* **69**(4), 925–942.

Legros, P. & Newman, A. F. (2007), 'Beauty is a beast, frog is a prince: Assortative matching with nontransferabilities', *Econometrica* **75**(4), 1073–1102.

Marx, P. & Schummer, J. (2019), 'Revenue from matching platforms'. Working Paper.

Shimer, R. & Smith, L. (2000), 'Assortative matching and search', *Econometrica* **68**(2), 343–369.

Tamura, W. (2016), 'Auction platform design and the linkage principle', *Journal of Industrial Economics* **64**(2), 201–225.

Appendix

The Appendix consists of three sections. Section A.1 studies a model of seller-offer bargaining and compares it with buyer-offer bargaining in the text. Section A.2 discusses the BNE of the trading game most preferred by the platform. A mode of one-sided competition is presented in Section A.3. All the proofs of the results presented in the text are collected in Section A.4.

A.1 Seller-Offer versus Buyer-Offer Bargaining

The analysis in the previous section studies the optimal matching mechanism under the pricing rule $k(z) = z_B$ which corresponds to buyer-offer bargaining. A natural question concerns whether or not the platform can do better by having the sellers make offers instead. Given the symmetric nature of the problems, we expect the answer to depend on the proportion of types on each side. We begin by describing the optimal mechanism with seller-offer bargaining, which is expressed formally by setting the pricing rule $k(z) = z_A$. Let $\sigma = (\sigma_A, \sigma_B)$ be the strategy profile of this game in which a buyer plays a weakly dominant strategy of bidding his own type: $\sigma_B(\beta) = \beta$ for every β. Given his belief $\tilde{p}_B$ about a buyer's type, a seller's optimal bid is then given by

$$z_A^*(\alpha, \tilde{p}_B) = \begin{cases} \beta_1 & \text{if } \alpha = \alpha_1 \text{ and } \tilde{p}_B(\beta_2) < \frac{\gamma}{1+\gamma}, \\ \beta_2 & \text{if } \alpha = \alpha_2, \text{ or if } \alpha = \alpha_1 \text{ and } \tilde{p}_B(\beta_2) \geq \frac{\gamma}{1+\gamma}. \end{cases}$$

Again, the only viable bid for the high-cost (α_2) seller is β_2, whereas the optimal bid for the low-cost (α_1) seller is either high or low depending on his belief about the type of the matched buyer. We specify the seller's strategy σ_A by letting $\sigma_A(\alpha) = z_A^*(\alpha, p_B(\cdot \mid \alpha))$ for every α. The assortative and random matching rules are as defined in the previous section.[18] As a counterpart to the B-squeeze matching rule defined in the previous section, when $\lambda_1 + (1+\gamma)\mu_1 \geq 1$, we define a matching rule p to be *S-squeeze* (SSM) if

$$(p_{11}, p_{12}, p_{21}, p_{22}) = \Big(\mu_1 - \frac{1}{1+\gamma}\,\lambda_2,\ \mu_2 - \frac{\gamma}{1+\gamma}\,\lambda_2,\ \frac{1}{1+\gamma}\,\lambda_2,\ \frac{\gamma}{1+\gamma}\,\lambda_2\Big).$$

We can verify that under SSM, a low-cost seller α_1 is matched more often with a high-valuation buyer β_2 than a high-cost seller α_2 is wherever it is relevant (*i.e.*, $\lambda_1 + (1+\gamma)\mu_1 \geq 1$ and $\mu_1 \leq \frac{1}{1+\gamma}$). This rule also makes a low-cost seller α_1 exactly indifferent between bidding high β_2 and bidding low β_1 when he faces the type distribution of a buyer intended for a high-cost seller α_2, and minimizes the informational rent of the low-cost seller by the same logic as in the case of B-squeeze matching.

Proposition A.1 *Suppose that the mechanism Γ is optimal with seller-offer bargaining. Then its matching rule p and revenue $R(\Gamma)$ are given as follows.*

i. If $\lambda_1 + (1+\gamma)\mu_1 \leq 1$, then p is PAM and $R(\Gamma) = (1+\gamma)\mu_2 - \frac{\mu_2 - \lambda_1}{\lambda_2}$.

ii. If $\lambda_1 \leq \frac{1}{1+\gamma}$, $\gamma\lambda_1 + \mu_1 \leq 1$, and $\lambda_1 + (1+\gamma)\mu_1 > 1$, then p is PAM and

$$R(\Gamma) = \begin{cases} \frac{\mu_2 - \lambda_1}{\lambda_2}\,\gamma + \lambda_1 & \text{if } \lambda_1 \leq \mu_2, \\ (1+\gamma)\mu_2 - \lambda_1\gamma & \text{otherwise.} \end{cases}$$

iii. If $\lambda_1 > \frac{1}{1+\gamma}$, $\lambda_1 + (1+\gamma)\mu_1 > 1$, and $\mu_1 \leq \frac{1}{1+\gamma}$, then p is SSM and $R(\Gamma) = \mu_2(1+\gamma) - \frac{\gamma}{1+\gamma}$.

iv. If $\mu_1 > \frac{1}{1+\gamma}$ and $\gamma\lambda_1 + \mu_1 > \gamma$, then p is RM and $R(\Gamma) = \gamma\mu_2$.

As seen in Figure 6, the optimal configuration with seller-offer bargaining is exactly symmetric to that with buyer-offer bargaining with respect to the diagonal line $\lambda_1 + \mu_1 = 1$ ($\Leftrightarrow \lambda_1 = \mu_2$).

In general, comparison of performance between buyer-offer and seller-offer bargaining is not straightforward.[19] We can however verify that PAM with buyer-offer (resp. seller-offer) bargaining dominates RM with seller-offer (resp. buyer-offer) bargaining.

Proposition A.2 *Let Γ be an optimal mechanism with either seller-offer or buyer-offer bargaining. Then the associated matching rule p is either PAM, BSM, or SSM.*

---

[18] They can alternatively be defined in terms of $z = p_B(\beta_2 \mid \alpha_1)$ and $w = p_B(\beta_2 \mid \alpha_2)$: p is assortative if

$$(p_B(\beta_2 \mid \alpha_1),\, p_B(\beta_2 \mid \alpha_2)) = \begin{cases} \Big(1,\ \frac{\mu_2 - \lambda_1}{\lambda_2}\Big) & \text{if } \mu_2 > \lambda_1, \\ \Big(\frac{\mu_2}{\lambda_1},\ 0\Big) & \text{if } \mu_2 < \lambda_1, \end{cases}$$

and random if $(p_B(\beta_2 \mid \alpha_1),\, p_B(\beta_2 \mid \alpha_2)) = (\mu_2, \mu_2)$.

[19] In particular, it is difficult to establish the dominance relationship between PAM with buyer-offer (resp. seller-offer) bargaining and SSM (resp. BSM) with seller-offer (resp. buyer-offer) bargaining.

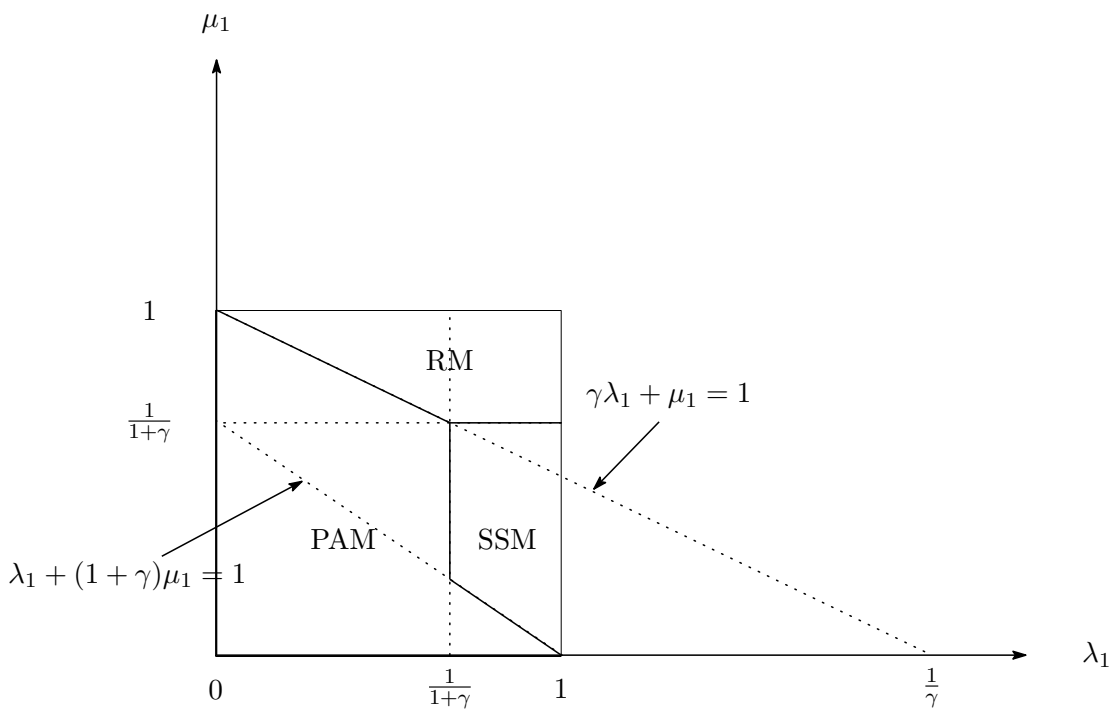

Figure 6: Optimal Matching with Seller-Offer Bargaining

We next consider a market that deviates from symmetry just slightly. We say that side A has higher (resp. lower) quality than side B if the proportion of low cost sellers on side A is higher (resp. lower) than that of high valuation buyers on side B: $\lambda_1 > \mu_2$. The following proposition shows that in a slightly asymmetric market, the optimal mechanism employs a protocol where the side with the lower quality makes an offer. Put differently, it is optimal to have seller-offer bargaining if the proportion of low-cost sellers on side A is lower than the proportion of high-valuation buyers on side B, and vice versa.

Proposition A.3 *Take any $d \in (0,1)$. There exists $\varepsilon > 0$ such that if $\|(\lambda_1, \mu_2) - (d,d)\| < \varepsilon$, then the optimal mechanism Γ entails buyer-offer bargaining if $\lambda_1 > \mu_2$ and seller-offer bargaining if $\lambda_1 < \mu_2$.*

A clear conclusion is possible regarding the comparison of buyer-offer and seller-offer bargaining when the market is symmetric.

Proposition A.4 *Suppose that the market is symmetric ($\lambda_1 = \mu_2$). Then the optimal mechanism with buyer-offer bargaining and that with seller-offer bargaining yield the same revenue.*

It follows from Corollary 4.1 and Proposition A.4 that either with buyer-offer or seller-offer bargaining, PAM is dominated by BSM or SSM when the market is symmetric and $\lambda_1 = \mu_2 > \frac{1}{1+\gamma}$.

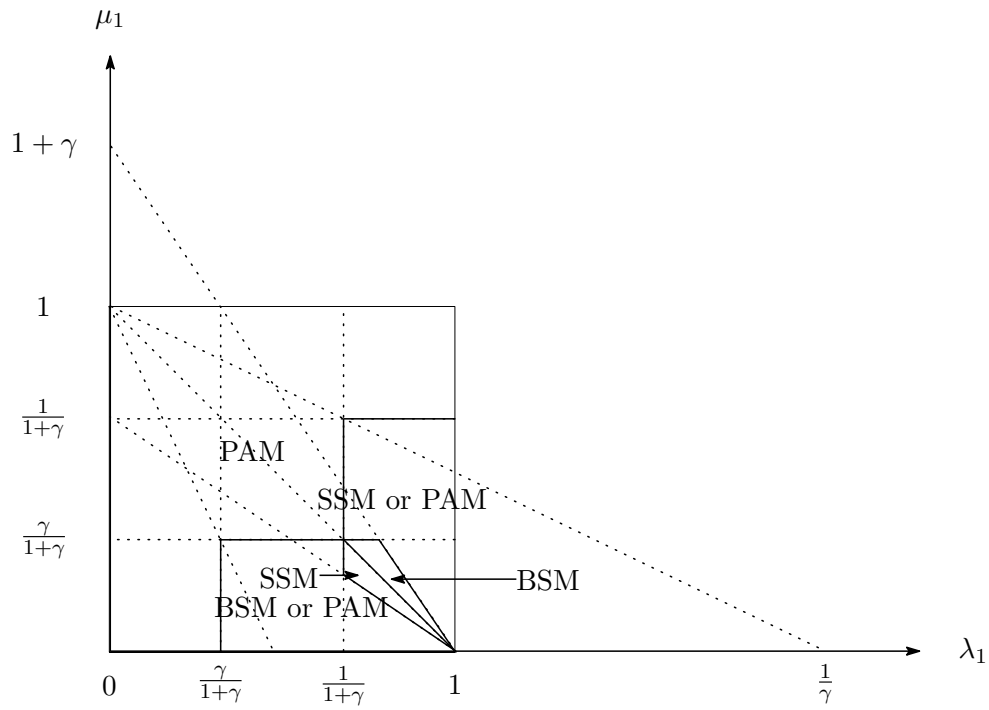

Figure 7: Optimal Matching with Seller-Offer or Buyer-Offer Bargaining

PAM: Positive assortative with buyer- or seller-offer
BSM: B-squeeze with buyer-offer
SSM: S-squeeze with seller-offer

A.2 Optimal Equilibrium of the Trading Game

Our analysis in the main text focuses on a weakly undominated BNE of a game of sequential moves with a buyer making a take-it-or-leave-it offer. We suppose in this section that the agents play a BNE σ that is most preferred by the platform.

Let the price equal $k(z) = kz_A + (1-k)z_B$ for a constant $k \in [0,1]$, and suppose that the mechanism specifies a strategy profile $\sigma = (\sigma_A, \sigma_B)$ such that

$$\sigma_A(\alpha) = \begin{cases} \zeta & \text{if } \alpha = \alpha_1, \\ \beta_2 & \text{if } \alpha = \alpha_2, \end{cases} \quad \text{and} \quad \sigma_B(\beta) = \zeta \quad \text{for every } \beta, \tag{14}$$

where $\zeta \in [\alpha_1, \beta_1]$. σ_B is the buyer's best response against σ_A regardless of his belief $\tilde{p}_A$ about α. Furthermore, σ_A is also the seller's best response against σ_B although the high cost type (α_2) will never trade his good under σ. The following proposition shows that when σ is as given in (14) for $\zeta = \alpha_1$, the platform's revenue in the symmetric market equals the first-best level identified in Proposition 3.1.

Proposition A.5 *Suppose that the mechanism Γ entails PAM and the BNE σ in (14) with $\zeta = \alpha_1$. When the market is symmetric, Γ extracts full surplus from the agents and hence is optimal.*

As noted above, σ in Proposition A.5 is not the most natural BNE when for example $k = 0$ (buyer-offer bargaining) or $k = 1$ (seller-offer bargaining): When $k = 0$, $\sigma_A(\alpha_2) = \beta_2$ is weakly dominated for the high-cost seller (α_2), and when $k = 1$, $\sigma_B(\beta) = \alpha_1$ is weakly dominated for both buyer types. In other words, it is not possible to replicate such a σ in a buyer-offer or seller-offer game while requiring sequential rationality.

A.3 Model of One-sided Competition

Suppose that side A has a mass of sellers indexed by numbers in $[0, 1]$, whereas side B has a mass of buyers indexed by numbers in $[0, 2]$. Each seller is endowed with a single unit of a good of quality α, which represents his type: The good is either high quality α_2 or low quality α_1.[20] Each buyer has a unit demand for the good, and has type β that reflects his valuation of the good: The type is either high β_2 or low β_1. For a buyer of type β, the value of the good of quality α is given by $v(\alpha, \beta)$. Denote

$$v_{11} = v(\alpha_1, \beta_1), \quad v_{12} = v(\alpha_1, \beta_2), \quad v_{21} = v(\alpha_2, \beta_1), \quad \text{and} \quad v_{22} = v(\alpha_2, \beta_2).$$

We assume that v has increasing differences in the sense that

$$0 \leq \Delta_1 \equiv v_{12} - v_{11} < v_{22} - v_{21} \equiv \Delta_2. \tag{15}$$

Equivalently, the marginal increase in utility in response to an increase in quality is higher for the high-valuation buyer than for the low-valuation buyer. The seller's valuation of the good equals zero regardless of its quality.[21] Each seller is type α_i with probability λ_i and each buyer is type β_i with probability μ_i. The type realizations are independent across agents so that we may again identify λ_i as the proportion of type α_i sellers in side A, and μ_i as the proportion of type β_i buyers on side B. We assume that when matched with a seller, buyers observe the quality α of the seller's good.

Matching between a seller and two buyers is implemented through the allocation of buyers to two buyer slots: A buyer with index $k \leq 1$ is allocated to the first slot while a buyer with index $k > 1$ is allocated to the second slot.[22] A matching rule $p = (p_{111}, \ldots, p_{222})$ is a probability distribution over $A \times B^2$: p_{ijk} (i, j, $k \in \{1, 2\}$) is the probability that any given match involves a seller of type α_i along with a buyer of type β_j in the first slot, and a buyer of type β_k in the second slot. We assume that the platform treats the two buyer slots symmetrically:

$$p_{ijk} = p_{ikj} \quad \text{for any } i,\ j,\ k \in \{1, 2\}. \tag{16}$$

The matching rule p must also be consistent with the type distribution in the population:

$$\begin{aligned} p_{111} + 2p_{112} + p_{122} = \lambda_1 \quad &\Leftrightarrow \quad p_{211} + 2p_{212} + p_{222} = \lambda_2, \\ p_{111} + p_{112} + p_{211} + p_{212} = \mu_1 \quad &\Leftrightarrow \quad p_{121} + p_{122} + p_{221} + p_{222} = \mu_2. \end{aligned} \tag{17}$$

[20] Uncertainty about the quality of the sellers' good is also assumed in Tamura (2016).

[21] It follows that the aggregate surplus from trade in a match involving seller of type α_i and buyers of types β_j and β_k can be written as $f(\alpha_i, \beta_j, \beta_k) = \max\{v_{ij}, v_{ik}\}$. (15) is not consistent with the supermodularity of f which would require $v_{12} - v_{11} = v_{22} - v_{21} = 0$ when the ordering on the domain of f is induced by $\alpha_1 \prec \alpha_2$ and $\beta_1 \prec \beta_2$.

[22] Given the independence of type realizations, we may assume that the type distributions of buyers are the same between the first and second intervals.

(17) is a version of the Bayes plausibility conditions when every seller is matched with two buyers. The set P of feasible matching rules is hence given by

$$P = \{p \in A \times B^2 : p \text{ satisfies (16) and (17)}\}.$$

We suppose that each seller sells his good to the matched buyers through a first-price auction: Each buyer submits a sealed bid and pays the price equal to his bid in the case of winning. Consider now the game on the path where both buyers have reported their types truthfully. In this game, the joint distribution of type profiles is given by p. Let $\sigma_B(\beta \mid \alpha)$ denote the BNE bidding strategy of a type β buyer in this game when the quality of the seller's good equals α. By the standard argument, a low valuation buyer (β_1) bids his value $v_{\alpha 1}$: $\sigma_B(\beta_1 \mid \alpha) = v_{\alpha 1}$. On the other hand, a high valuation buyer (β_2) randomizes his bid. The cumulative distribution G_α of β_2's random bid has support $[v_{\alpha 1}, \bar{b}_\alpha]$ for some $\bar{b}_\alpha$. In the Appendix, we show that $\bar{b}_\alpha$ and G_α are given by

$$\bar{b}_\alpha = \Pr(\beta_1 \mid \alpha, \beta_2) v_{\alpha 1} + \Pr(\beta_2 \mid \alpha, \beta_2) v_{\alpha 2},$$

and

$$G_\alpha(b) = \frac{\Pr(\beta_1 \mid \alpha, \beta_2)}{\Pr(\beta_2 \mid \alpha, \beta_2)} \left(\frac{b - v_{\alpha 1}}{v_{\alpha 2} - b} \right) = \frac{p_{\alpha 12}}{p_{\alpha 22}} \left(\frac{b - v_{\alpha 1}}{v_{\alpha 2} - b} \right).$$

As seen, a high-valuation buyer's BNE bidding strategy $\sigma(\beta_2 \mid \alpha)$ depends on his belief about the type of the other buyer in his match. Specifically, when his belief places more weight on the other buyer being also the high-valuation type (β_2), the distribution of his bid is higher in the sense of stochastic dominance.

To solve for the optimal matching mechanism, we first note that since a type β_2 bidder is indifferent over bids in the support of G_α, his BNE payoff in the game with a type α seller is given by $\Pr(\beta_1 \mid \alpha, \beta_2)\,(v_{\alpha 2} - v_{\alpha 1})$, which he would obtain by bidding slightly above $v_{\alpha 1}$. When he misreports, his expected payoff is likewise given by $\Pr(\beta_1 \mid \alpha, \beta_1)\,(v_{\alpha 2} - v_{\alpha 1})$. For a type β_1 seller, his expected payoff equals zero whether or not he reports truthfully. These considerations give rise to the incentive conditions for the buyer. On the other hand, a seller's expected payoff after truthful reporting as well as misreporting can be computed from the expected payment by a buyer.[23]

Note that PAM in this model would create a pair of high-valuation buyers as much as possible, and also match a high-quality seller with a pair of high-valuation buyers as much as possible. It turns out, however, that the optimal mechanism does not entail PAM in this sense. Instead, it is a variation of assortative matching as follows. We say that p entails *negative assortative matching* (NAM) *between buyers* if it matches a high type buyer with a low type buyer as much as possible, and vice versa:

$$p \in P^{\text{NAMBB}} \equiv \operatorname{argmax} \{\hat{p}_B(\beta_1 \mid \beta_2) : \hat{p} \in P\},$$

where $p_B(\beta_j \mid \beta_k) = \frac{\sum_i p_{ijk}}{\sum_{i,j'} p_{ij'k}}$ is the probability that a type β_k buyer is matched with a type β_j buyer. Next, p is *PAM between a seller and a buyer pair subject to NAM between buyers* if among those rules in P^{NAMBB}, it first maximizes the probability that a type α_2 seller is matched with the

[23] The buyer's IC and IR conditions are described by (37) and (38) in the Appendix whereas the corresponding conditions for the seller are given by (41) and (42).

buyer type pair (β_2, β_2), and then maximizes the probability that α_2 is matched with the buyer type pair (β_1, β_2):

$$p \in \operatorname{argmax}\{\hat{p}_{BB}(\beta_1, \beta_2 \mid \alpha_2) : \hat{p} \in P^1\}, \text{ where } P^1 \equiv \operatorname{argmax}\{\hat{p}_{BB}(\beta_2, \beta_2 \mid \alpha_2) : \hat{p} \in P^{\text{NAMBB}}\}.$$

The following proposition shows that the optimal matching rule combines PAM and NAM in the sense described above.

Proposition A.6 *Suppose that the good is traded through a first-price auction. Then the matching rule p under the optimal mechanism Γ entails PAM between a seller and a buyer pair subject to NAM between buyers.*

The intuition behind NAM between buyers is as follows. As before, the IC condition for a type β_2 buyer is binding so that

$$\begin{aligned}\pi_B(\sigma, \beta_2) - t_B(\beta_2) &= \sum_{\alpha} \Big\{ \Pr(\alpha, \beta_1 \mid \beta_1)\,(v_{\alpha 2} - v_{\alpha 1}) + \Pr(\alpha, \beta_2 \mid \beta_1) \cdot 0 \Big\} - t_B(\beta_1), \\ &= \sum_{\alpha} \Pr(\alpha, \beta_1 \mid \beta_1)\,\Delta_\alpha - t_B(\beta_1),\end{aligned}$$

where the right-hand side is β_2's payoff when he misreports his type as β_1. It follows that β_2's informational rent is minimized when $p_B(\beta_1 \mid \beta_1) = \sum_\alpha \Pr(\alpha, \beta_1 \mid \beta_1)$ is minimized, or equivalently, when $p_B(\beta_2 \mid \beta_1)$ is maximized as entailed by NAM between buyers.

Suppose now that the platform instead uses the second-price auction as the trading game. It is then a weakly dominant strategy for a buyer to bid his true valuation. We hence specify σ_B as

$$\sigma_B(\beta \mid \alpha) = v(\alpha, \beta) \quad \text{for every } (\alpha, \beta).^{24}$$

Unlike in the first-price auction, a buyer's belief does not influence his bidding behavior. As can be verified, however, the expected payoff of a buyer of each type is equal to that under the first-price auction, and so is the seller's expected payoff. It follows that the platform's problem is unchanged, and the optimal matching mechanism under the second price auction is the same as that under the first-price auction.

Proposition A.7 *Suppose that the good is traded through the second-price auction. The matching rule p under the optimal matching mechanism Γ is the same as that in Proposition A.6 for the first-price auction.*

We next examine the welfare implication of the optimal matching mechanism Γ identified in Proposition A.6. Note that the efficiency of Γ is expressed in terms of its matching rule p by

$$W(p) = p_{111}v_{11} + (2p_{112} + p_{122})v_{12} + p_{211}v_{21} + (2p_{212} + p_{222})v_{22}.$$

[24] Denote by $f_B(\beta_j; \alpha, \beta_k)$ the value of a match to a buyer in this BNE when his own type is β_j, the other bidder's type is β_k, and the seller's type is α. The BNE value of a match to agents is not supermodular. In fact, when a buyer's type changes from β_1 to β_2, the difference in his payoff against (α_2, β_2) is smaller than that against (α_1, β_1):

$$f_B(\beta_1; \alpha_2, \beta_2) - f_B(\beta_1; \alpha_1, \beta_1) = 0 > -(v_{12} - v_{11}) = f_B(\beta_2; \alpha_2, \beta_2) - f_B(\beta_2; \alpha_1, \beta_1).$$

Proposition A.8 *The optimal mechanism* Γ *of the auction platform is efficient: If* p *is the matching rule in the optimal mechanism* Γ*, then* $p \in \operatorname{argmax}\{W(\hat{p}) : \hat{p} \in P\}$.

The first-best efficiency of the auction platform can be understood as follows. It is efficient for a high-valuation buyer to win the object whenever possible, and it is not efficient to match two high-valuation buyers when it can be avoided since then one of them must lose out. The NAM property of the optimal matching rule suggests that the optimal mechanism also avoids the head-to-head encounter of high-valuation buyers as much as possible. Subject to this, however, it is optimal for the high-quality good to go to a high-valuation buyer, which is also implied by the PAM property of the optimal matching between sellers and buyer pairs. The optimal matching is hence aligned with first-best efficiency.

A.4 Proofs

For simplicity, we use the following notation in the analysis of the trading platform in the Appendix:

$$\begin{aligned} x &= p_A(\alpha_1 \mid \beta_1) = \tfrac{p_{11}}{\mu_1}, \quad y = p_A(\alpha_1 \mid \beta_2) = \tfrac{p_{12}}{\mu_2}, \\ z &= p_B(\beta_2 \mid \alpha_1) = \tfrac{p_{12}}{\lambda_1}, \quad w = p_B(\beta_2 \mid \alpha_2) = \tfrac{p_{22}}{\lambda_2}. \end{aligned} \tag{18}$$

Proof of Proposition 3.1. The mechanism is efficient only if $\alpha < \beta$ implies $\sigma_A(\alpha) \leq \sigma_B(\beta)$ so that transaction takes place with probability one between any such pair of agents. In this case, the social welfare W is described as

$$\begin{aligned} W &= \gamma\{p_{11} + p_{22}\} + (1+\gamma)p_{12} \\ &= \gamma\{\mu_1 x + \mu_2(1-y)\} + (1+\gamma)\mu_2 y \\ &= \gamma\mu_1 x + \mu_2 y + \gamma\mu_2. \end{aligned}$$

Since $\frac{\mu_1}{\mu_2} > \frac{\gamma\mu_1}{\mu_2}$, maximization of W with respect to (x, y) subject to the Bayes plausibility condition (5) implies that y should be maximized subject to it. This shows that p is PAM. Substitution of (x, y) for PAM in (8) yields (9). ■

Proof of Proposition 3.2. Define as in (18) $x = p_A(\alpha_1 \mid \beta_1)$ and $y = p_A(\alpha_1 \mid \beta_2)$. The IC and IR conditions for a type β_1 buyer can be written as:

$$xk(\beta_1 - \alpha_1) - t_B(\beta_1) \geq \max\{0,\, yk(\beta_1 - \alpha_1) - t_B(\beta_2)\},$$

and those for a type β_2 buyer can be written as:

$$yk(\beta_2 - \alpha_1) + (1-y)k(\beta_2 - \alpha_2) - t_B(\beta_2) \geq \max\{0,\, xk(\beta_2 - \alpha_1) + (1-x)k(\beta_2 - \alpha_2) - t_B(\beta_1)\}.$$

Since these imply

$$(y-x)k(\beta_1 - \alpha_1) \leq t_B(\beta_2) - t_B(\beta_1) \leq (y-x)k(\alpha_2 - \alpha_1),$$

we need $y \geq x$ for the feasibility of the mechanism. In this case, the optimal transfers are given by

$$t_B(\beta_1) = xk\gamma \quad \text{and} \quad t_B(\beta_2) = t_B(\beta_1) + (y-x)k.$$

On the other hand, the IC and IR conditions for a type α_1 seller are given by

$$(1-z)(1-k)(\beta_1-\alpha_1)+z(1-k)(\beta_2-\alpha_1)-t_A(\alpha_1) \geq \max\{0,\ (1-w)(1-k)(\beta_1-\alpha_1)+w(1-k)(\beta_2-\alpha_1)-t_A(\alpha_2)\},$$

and those for a type α_2 seller are given by

$$w(1-k)(\beta_2-\alpha_2)-t_A(\alpha_2) \geq \max\{0,\ z(1-k)(\beta_2-\alpha_2)-t_A(\alpha_1)\}.$$

Together, these imply

$$(z-w)(1-k)(\beta_2-\alpha_2) \leq t_A(\alpha_1)-t_A(\alpha_2) \leq (z-w)(1-k)(\beta_2-\beta_1),$$

and hence feasibility requires $z \geq w$, or equivalently, $y \geq \lambda_1$.[25] In this case, the optimal transfers are given by

$$t_A(\alpha_2) = \frac{\mu_2}{\lambda_2}(1-y)(1-k)\gamma \quad \text{and} \quad t_A(\alpha_1) = t_A(\alpha_2) + \left(\frac{\mu_2}{\lambda_1}y - \frac{\mu_2}{\lambda_2}(1-y)\right)(1-k).$$

It follows that the platform's revenue from both sides of the market under the optimal transfer functions is given by

$$\begin{aligned} R(\Gamma) &= w(1-k)\gamma + \lambda_1(z-w)(1-k) + xk\gamma + \mu_2(y-x)k \\ &= \frac{\mu_2}{\lambda_2}(1-y)(1-k)\gamma + \frac{\mu_2}{\lambda_2}(y-\lambda_1)(1-k) + xk\gamma + \mu_2(y-x)k \\ &= k(\gamma-\mu_2)x + \frac{\mu_2}{\lambda_2}\left\{(1-k)(1-\gamma)+k\lambda_2\right\}y + \frac{\mu_2}{\lambda_2}(1-k)(\gamma-\lambda_1). \end{aligned}$$

Note that the following relationship holds between the gradient vector (μ_1, μ_2) of the Bayes plausibility condition (5) for x and y, and the gradient vector of R above:

$$\frac{\mu_1}{\mu_2} > \frac{k(\gamma-\mu_2)}{\frac{\mu_2}{\lambda_2}\left\{(1-k)(1-\gamma)+k\lambda_2\right\}}.$$

This implies that the maximization of R entails the maximization of y subject to Bayes plausibility (5), and the feasibility constraints $y \geq x$ and $y \geq \lambda_1$. Therefore, the optimal matching rule p is assortative. When $\lambda_1 \geq \mu_2$, substitution of $x = \frac{\lambda_1-\mu_2}{\mu_1}$ and $y = 1$ yields the maximized revenue as in the first line of (11), and when $\lambda_1 < \mu_2$, substitution of $x = 0$ and $y = \frac{\lambda_1}{\mu_2}$ yields the maximized revenue as in the second line of (11). ■

Proof of Proposition 4.1. We proceed by separating cases based on the buyer's belief about the seller's type induced by the matching rule p.

1. The optimal bid for the high-valuation buyer β_2 is α_1 when he has reported type β_2 truthfully, and also when he has misreported his type to be β_1:

$$z_B^*(\beta_2, p_A(\cdot \mid \beta)) = \alpha_1 \text{ for any } \beta \in B.$$

[25] This holds since $z = \frac{\mu_2}{\lambda_1}y$ and $w = \frac{\mu_2}{\lambda_2}(1-y)$.

This requires that x, $y \geq \frac{\gamma}{1+\gamma}$. In this case, Bayes plausibility implies that the proportion of the low-cost seller must be high in the population:

$$\lambda_1 = \Pr(\alpha_1) = \mu_1 x + \mu_2 y \geq \frac{\gamma}{1+\gamma}.$$

The IC and IR conditions for a type β_1 buyer are written as

$$\begin{aligned} &p_A(\alpha_1 \mid \beta_1)(\beta_1 - \alpha_1) + p_A(\alpha_2 \mid \beta_1) \cdot 0 - t_B(\beta_1) \\ &\geq \max\{0,\, p_A(\alpha_1 \mid \beta_2)(\beta_1 - \alpha_1) + p_A(\alpha_2 \mid \beta_2) \cdot 0 - t_B(\beta_2)\}. \end{aligned} \tag{19}$$

Note that the left-hand side is his expected payoff when he reports β_1: The first term corresponds to the event that he is matched against a low-cost seller so that his offer α_1 will be accepted and trade takes place. The second term correspond to the event that he is matched against a high-cost seller so that his offer will be rejected and no trade takes place. The right-hand side is the maximum between the buyer's reservation payoff and his expected payoff when he reports β_2. The IC and IR conditions for a type β_2 buyer are similarly given by

$$\begin{aligned} &p_A(\alpha_1 \mid \beta_2)(\beta_2 - \alpha_1) + p_A(\alpha_2 \mid \beta_1) \cdot 0 - t_B(\beta_2) \\ &\geq \max\{0,\, p_A(\alpha_1 \mid \beta_2)(\beta_2 - \alpha_1) + p_A(\alpha_2 \mid \beta_2) \cdot 0 - t_B(\beta_1)\}. \end{aligned} \tag{20}$$

Using the short-hand notation introduced in (18), we can summarize (19) and (20) as

$$\begin{aligned} &(y - x)(\beta_1 - \alpha_1) \leq t_B(\beta_2) - t_B(\beta_1) \leq (y - x)(\beta_2 - \alpha_1), \\ &t_B(\beta_1) \leq x(\beta_1 - \alpha_1), \\ &t_B(\beta_2) \leq y(\beta_2 - \alpha_1). \end{aligned}$$

This is feasible if

$$y = p_A(\alpha_1 \mid \beta_2) \geq p_A(\alpha_1 \mid \beta_1) = x, \tag{21}$$

and the optimal transfer function t_B is given by

$$t_B(\beta_1) = x(\beta_1 - \alpha_1) \quad \text{and} \quad t_B(\beta_2) = t_B(\beta_1) + (y - x)(\beta_2 - \alpha_1).$$

Turning now to side A, we note that the seller's payoff in the trading game equals zero regardless of his type since both buyer types bid α_1 under q. It follows that the only transfer function t_A that satisfies IC and IR for the seller is given by $t_A(\alpha_1) = t_A(\alpha_2) = 0$. The platform's revenue is then given by

$$\begin{aligned} R(\Gamma) &= \mu_1 t_B(\beta_1) + \mu_2 t_B(\beta_2) \\ &= x(\beta_1 - \alpha_1) + \mu_2(y - x)(\beta_2 - \alpha_1) \\ &= x\{\gamma - \mu_2(1+\gamma)\} + y\mu_2(1+\gamma). \end{aligned}$$

Since R is linear in x and y, comparison of their coefficients against those in the Bayes plausibility condition $\mu_1 x + \mu_2 y = \lambda_1$ determines the optimal matching rule. Specifically, since

$$\frac{\mu_1}{\mu_2} > \frac{\gamma - \mu_2(1+\gamma)}{\mu_2(1+\gamma)} \quad \Leftrightarrow \quad 1 + \gamma > \gamma,$$

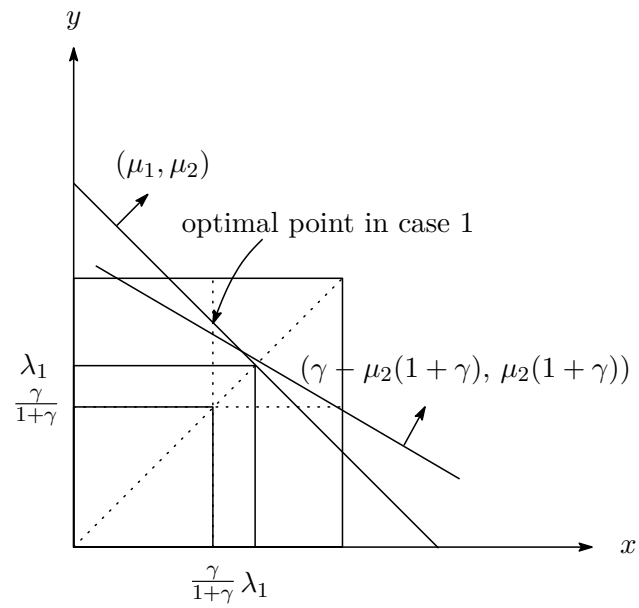

Figure 8: Optimal choice of (x, y) in case 1

the optimal p should maximize y subject to the feasibility constraints: $x,\ y \geq \frac{\gamma}{1+\gamma}$, $y \geq x$ and $\mu_1 x + \mu_2 y = \lambda_1 \geq \frac{\gamma}{1+\gamma}$. As seen in Figure 8, this yields

$$(x, y) = \begin{cases} \left(\frac{\gamma}{1+\gamma},\ \frac{\lambda_1}{\mu_2} - \frac{\mu_1}{\mu_2}\frac{\gamma}{1+\gamma}\right) & \text{if } (1+\gamma)\lambda_1 + \mu_1 \leq 1+\gamma, \\ \left(\frac{\lambda_1 - \mu_2}{\mu_1},\ 1\right) & \text{if } (1+\gamma)\lambda_1 + \mu_1 > 1+\gamma. \end{cases}$$

The maximized revenue is given by

$$R^* = \begin{cases} \lambda_1(1+\gamma) - \frac{\gamma}{1+\gamma} & \text{if } (1+\gamma)\lambda_1 + \mu_1 \leq 1+\gamma, \\ \gamma\lambda_1 + \frac{\mu_2}{\mu_1}\lambda_2 & \text{if } (1+\gamma)\lambda_1 + \mu_1 > 1+\gamma. \end{cases}$$

2. The optimal bid for the buyer of type β_2 equals α_1 when he reports his type truthfully, but α_2 when he misreports his type to be β_1. This requires $x \leq \frac{\gamma}{1+\gamma} \leq y$.

   The IC and IR constraints of the type β_2 buyer are given by

$$y(\beta_2 - \alpha_1) - t_B(\beta_2) \geq \max\{0,\ \beta_2 - \alpha_2 - t_B(\beta_1)\}.$$

   The IC and IR constraints of the type β_1 buyer are given by

$$x(\beta_1 - \alpha_1) - t_B(\beta_1) \geq \max\{0,\ y(\beta_1 - \alpha_1) - t_B(\beta_2)\}.$$

   These can be summarized as:

$$\begin{aligned} &(y - x)\gamma \leq t_B(\beta_2) - t_B(\beta_1) \leq y(1+\gamma) - \gamma, \\ &t_B(\beta_1) \leq x\gamma, \\ &t_B(\beta_2) \leq y(1+\gamma). \end{aligned}$$

For this to be feasible, we need

$$(y-x)\gamma \leq y(1+\gamma)-\gamma \quad \Leftrightarrow \quad \gamma x + y \geq \gamma. \tag{22}$$

Furthermore, there exists (x, y) that satisfies $0 \leq x \leq \frac{\gamma}{1+\gamma} \leq y \leq 1$, $\gamma x + y \geq \gamma$ and Bayes plausibility (5) if and only if

$$\begin{aligned} &(1+\gamma)\lambda_1 + \mu_1 \leq 1+\gamma, \text{ and} \\ &\text{either } \lambda_1 \geq \frac{\gamma}{1+\gamma} \text{ or } \lambda_1 + \gamma\mu_1 \geq \gamma. \end{aligned} \tag{23}$$

The optimal transfer function t_B for the buyer is then given by

$$t_B(\beta_1) = x\gamma \quad \text{and} \quad t_B(\beta_2) = t_B(\beta_1) + y(1+\gamma) - \gamma.$$

The seller's payoff in the trading game equals zero since both buyer types bid α_1 according to q. It follows that the transfer function for the seller equals $t_A(\alpha_1) = t_A(\alpha_2) = 0$, and that the platform's revenue is given by

$$R(\Gamma) = x\gamma + y\mu_2(1+\gamma) - \gamma\mu_2.$$

The optimal matching rule p maximizes R subject to $x \leq \frac{\gamma}{1+\gamma} \leq y$, (22), and Bayes plausibility $\mu_1 x + \mu_2 y = \lambda_1$. In what follows, we separate cases depending on the values of λ_1 and μ_1. For this, it is useful to note that

$$\mu_1 < \frac{\gamma}{1+\gamma} \quad \Leftrightarrow \quad \gamma > \frac{\mu_1}{\mu_2} \quad \Leftrightarrow \quad \frac{\gamma}{(1+\gamma)\mu_2} > \frac{\mu_1}{\mu_2}, \tag{24}$$

where the second term corresponds to the comparison between the normal vectors of (22) and Bayes plausibility, and the third term corresponds to the comparison between the normal vectors of the revenue function R and Bayes plausibility.

(a) $\lambda_1, \mu_1 < \frac{\gamma}{1+\gamma}$. There is no (x, y) that satisfies $x \leq \frac{\gamma}{1+\gamma} \leq y$, $\gamma x + y \geq \gamma$, and $\mu_1 x + \mu_2 y = \lambda_1$. No p hence satisfies feasibility in this case.

(b) $\mu_1 < \frac{\gamma}{1+\gamma} < \lambda_1$. By (24), x should be as large as possible subject to feasibility, and the optimal matching rule p is such that

$$(x, y) = \left(\frac{\gamma}{1+\gamma}, \frac{\lambda_1}{\mu_2} - \frac{\mu_1}{\mu_2}\frac{\gamma}{1+\gamma}\right).$$

The maximized revenue is given by

$$R^* = (1+\gamma)\lambda_1 - \frac{\gamma}{1+\gamma}.$$

(c) $\mu_1 > \frac{\gamma}{1+\gamma}$. By (24), y should be as large as possible subject to feasibility, and the optimal matching rule p is such that

$$(x, y) = \begin{cases} \left(\frac{\lambda_1 - \mu_2}{\mu_1}, 1\right) & \text{if } \lambda_1 > \mu_2, \\ \left(0, \frac{\lambda_1}{\mu_2}\right) & \text{if } \lambda_1 < \mu_2. \end{cases}$$

The maximized revenue is give by

$$R^* = \begin{cases} \frac{\lambda_1 - \mu_2}{\mu_1}\gamma + \mu_2 & \text{if } \lambda_1 \geq \mu_2, \\ (1+\gamma)\lambda_1 - \mu_2\gamma & \text{otherwise.} \end{cases}$$

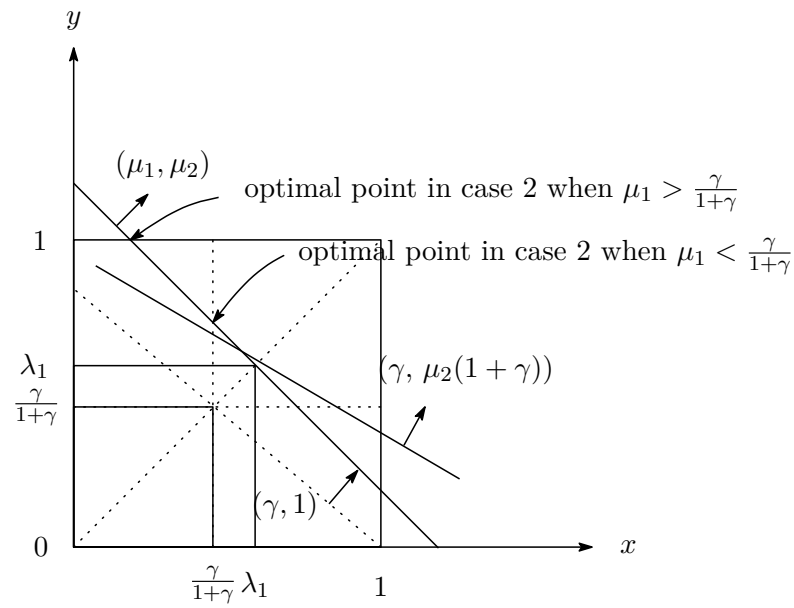

Figure 9: Optimal choice of (x, y) in case 2

Figure 9 illustrates the optimal matching rules when $\lambda_1 > \frac{\gamma}{1+\gamma}$.

3. The optimal bid for the type β_2 buyer is α_2 whether he has reported his type truthfully or not. This requires $x,\, y \leq \frac{\gamma}{1+\gamma}$.

   In this case, Bayes plausibility implies that the proportion of the type α_1 seller is low in the population:

$$\lambda_1 = \mu_1 x + \mu_2 y \leq \frac{\gamma}{1+\gamma}.$$

   The IC and IR constraints for a type β_1 buyer are given by

$$x\gamma - t_B(\beta_1) \geq \max\{0,\, y\gamma - t_B(\beta_2)\},$$

   and those for a type β_2 buyer are given by

$$\gamma - t_B(\beta_2) \geq \max\{0, \gamma - t_B(\beta_1)\}.$$

   These can be summarized as:

$$\begin{aligned}
&(y-x)\gamma \leq t_B(\beta_2) - t_B(\beta_1) \leq 0,\\
&t_B(\beta_1) \leq x\gamma,\\
&t_B(\beta_2) \leq \gamma.
\end{aligned}$$

   This is hence feasible if $y \leq x$. In this case, the optimal transfer function is given by

$$t_B(\beta_1) = t_B(\beta_2) = x\gamma.$$

   On the other hand, the IC and IR constraints for the type α_1 seller are given by

$$p_B(\beta_2 \mid \alpha_1)(\alpha_2 - \alpha_1) - t_A(\alpha_1) \geq \max\{0,\, p_B(\beta_2 \mid \alpha_2)(\alpha_2 - \alpha_1) - t_A(\alpha_2)\},$$

and those for the type α_2 seller are given by

$$-t_A(\alpha_2) \geq \max\{0,\ -t_A(\alpha_1)\}.$$

These can be summarized as

$$\begin{aligned}
&0 \leq t_A(\alpha_1) - t_A(\alpha_2) \leq \{p_B(\beta_2 \mid \alpha_1) - p_B(\beta_2 \mid \alpha_2)\}(\alpha_2 - \alpha_1),\\
&t_A(\alpha_1) \leq p_B(\beta_2 \mid \alpha_1)(\alpha_2 - \alpha_1),\\
&t_A(\alpha_2) \leq 0.
\end{aligned}$$

For this to be feasible, we need

$$\begin{aligned}
p_B(\beta_2 \mid \alpha_1) - p_B(\beta_2 \mid \alpha_2) \geq 0 \quad &\Leftrightarrow \quad \frac{\mu_2}{\lambda_1} p_A(\alpha_1 \mid \beta_2) \geq \frac{\mu_2}{\lambda_2} p_A(\alpha_2 \mid \beta_2)\\
&\Leftrightarrow \quad y \geq \lambda_1.
\end{aligned}$$

In this case, the optimal transfer function is given by

$$t_A(\alpha_1) = \frac{\mu_2}{\lambda_1\lambda_2}(y - \lambda_1) \quad \text{and} \quad t_A(\alpha_2) = 0.$$

It follows that the platform's revenue equals

$$R(\Gamma) = \gamma x + \frac{\mu_2}{\lambda_2}(y - \lambda_1).$$

The optimal matching rule p maximizes this subject to $x,\ y \leq \frac{\gamma}{1+\gamma}$, $x \geq y \geq \lambda_1$, and $\mu_1 x + \mu_2 y = \lambda_1$. The last two conditions however show that the feasible p is unique and such that $x = y = \lambda_1$. Therefore, the maximized revenue is given by

$$R^* = \lambda_1\gamma.$$

4. The optimal bid for a type β_2 buyer is α_2 when he reports his type truthfully, but α_1 when he misreports. This requires $x \geq \frac{\gamma}{1+\gamma} \geq y$.

   The IC and IR constraints for the type β_1 buyer are given by

$$x\gamma - t_B(\beta_1) \geq \max\{0,\ y\gamma - t_B(\beta_2)\},$$

   and those for the type β_2 buyer are given by

$$\gamma - t_B(\beta_2) \geq \max\{0,\ x(1+\gamma) - t_B(\beta_1)\}.$$

   Since these imply

$$(y - x)\gamma \leq t_B(\beta_2) - t_B(\beta_1) \leq \gamma - x(1+\gamma),$$

   feasibility requires

$$x + \gamma y \leq \gamma.$$

   On the other hand, the IC and IR constraints for the type α_1 seller are given by

$$p_B(\beta_2 \mid \alpha_1)(\alpha_2 - \alpha_1) - t_A(\alpha_1) \geq \max\{0,\ p_B(\beta_2 \mid \alpha_2)(\alpha_2 - \alpha_1) - t_A(\alpha_2)\},$$

and those for the type α_2 seller are given by

$$-t_A(\alpha_2) \geq \max\{0,\, -t_A(\alpha_1)\}.$$

These together imply

$$0 \leq t_A(\alpha_1) - t_A(\alpha_2) \leq \{p_B(\beta_2 \mid \alpha_1) - p_B(\beta_2 \mid \alpha_2)\}(\alpha_2 - \alpha_1).$$

Feasibility requires

$$\begin{aligned} p_B(\beta_2 \mid \alpha_1) \geq p_B(\beta_2 \mid \alpha_2) \quad &\Leftrightarrow \quad \frac{\mu_2}{\lambda_1} p_A(\alpha_1 \mid \beta_2) \geq \frac{\mu_2}{\lambda_2} p_A(\alpha_2 \mid \beta_2) \\ &\Leftrightarrow \quad y \geq \lambda_1. \end{aligned}$$

Note that $x \geq \frac{\gamma}{1+\gamma} \geq y \geq \lambda_1$ and $\mu_1 x + \mu_2 y = \lambda_1$ imply that $x = y = \lambda_1 = \frac{\gamma}{1+\gamma}$. In other words, feasibility holds only if $\lambda_1 = \frac{\gamma}{1+\gamma}$, and the optimal matching rule p is given by $x = y = \lambda_1$.

Summarizing the four cases above, we can conclude:

- If $\lambda_1 \geq \frac{\gamma}{1+\gamma}$ and $(1+\gamma)\lambda_1 + \mu_1 > 1+\gamma$, then only case 1 is feasible, and the optimal matching in case 1 is PAM. Hence, PAM is optimal.
- If $\lambda_1 \geq \frac{\gamma}{1+\gamma}$ and $(1+\gamma)\lambda_1 + \mu_1 \leq 1+\gamma$, then both cases 1 and 2 are feasible. The optimal matching in case 1 is BSM, whereas the optimal matching in case 2 is PAM if $\mu_1 > \frac{\gamma}{1+\gamma}$, and BSM otherwise. Hence, BSM is optimal if $\mu_1 \leq \frac{\gamma}{1+\gamma}$, and comparison of the revenue shows that PAM is optimal if $\mu_1 > \frac{\gamma}{1+\gamma}$.
- If $\lambda_1 < \frac{\gamma}{1+\gamma}$ and $\lambda_1 + \gamma\mu_1 < \gamma$, then only case 3 is feasible, and the only feasible matching in case 3 is RM. Hence, RM is optimal.
- If $\lambda_1 < \frac{\gamma}{1+\gamma}$ and $\lambda_1 + \gamma\mu_1 \geq \gamma$, then both cases 2 and 3 are feasible. The optimal matching in case 2 is PAM, and optimal matching in case 3 is RM. Comparison of the revenue under these two rules shows that PAM is optimal.

This completes the proof. ■

Proof of Proposition 4.2. As in the proof of Proposition 4.1, we separate cases depending on the values of x and y. Note that $(x, y) = \left(\frac{p_{11}}{\mu_1}, \frac{p_{12}}{\mu_2}\right)$.

1. $x,\ y > \frac{\gamma}{1+\gamma}$. In this case, a type β_2 buyer bids α_1 according to the BNE: $\sigma_B(\beta_2) = \alpha_1$. Social welfare is hence given by $W(\Gamma) = \gamma p_{11} + (1+\gamma)p_{12} = \gamma\frac{x}{\mu_1} + (1+\gamma)\frac{y}{\mu_2}$. The proof of Proposition 4.1 shows that there exists an IC and IR mechanism if and only if $y \geq x$. $W(\Gamma)$ is maximized when y is maximized subject to $y \geq x$ and Bayes plausibility (5). If follows that p is PAM.

2. $x \leq \frac{\gamma}{1+\gamma} \leq y$. A type β_2 buyer bids α_1 according to the BNE, and hence social welfare is again given by $W(\Gamma) = \gamma\frac{x}{\mu_1} + (1+\gamma)\frac{y}{\mu_2}$. The proof of Proposition 4.1 shows that there exists an IC and IR mechanism if and only if $\gamma x + y \geq \gamma$. The problem hence reduces to:

$$\max_{x,y} \gamma\frac{x}{\mu_1} + (1+\gamma)\frac{y}{\mu_2} \quad \text{subject to } \gamma x + y \geq \gamma,\ x \leq \frac{\gamma}{1+\gamma} \leq y, \text{ and Bayes plausibility (5).}$$

A feasible (x, y) exists if and only if $(1+\gamma)\lambda_1+\mu_1 \leq 1+\gamma$, and either $\lambda_1 \geq \frac{\gamma}{1+\gamma}$ or $\lambda_1+\gamma\mu_1 \geq \gamma$.

(a) If $\gamma \leq \frac{\lambda_1}{\mu_2}$ ($\Leftrightarrow \lambda_1 + \gamma\mu_1 \geq \gamma$), then PAM satisfies the constraints and maximizes W.

(b) If $\gamma > \frac{\lambda_1}{\mu_2}$ ($\Leftrightarrow \lambda_1 + \gamma\mu_1 < \gamma$), then a feasible (x, y) exists only if $\lambda_1 \geq \frac{\gamma}{1+\gamma}$. W is maximized when $\gamma x + y = \gamma$. Solving this and (5) simultaneously, we obtain

$$(x, y) = \left(\frac{\gamma\mu_2 - \lambda_1}{\gamma - (1+\gamma)\mu_1}, \frac{\gamma(\lambda_1 - \mu_1)}{\gamma - (1+\gamma)\mu_1}\right), \tag{25}$$

and

$$W(\Gamma) = \frac{\gamma\{\gamma\mu_2\lambda_1 - \lambda_1\mu_1 + \mu_2(\lambda_1 - \mu_1)\}}{\gamma - (1+\gamma)\mu_1}.$$

3. $x,\ y \leq \frac{\gamma}{1+\gamma}$. The proof of Proposition 4.1 shows that RM is the only feasible matching rule.

4. $x \geq \frac{\gamma}{1+\gamma} \geq y$. The proof of Proposition 4.1 shows that there exists no feasible matching rule.

When $(1+\gamma)\lambda_1+\mu_1 > 1+\gamma$, only case 1 is feasible and PAM is optimal. When $(1+\gamma)\lambda_1+\mu_1 \leq 1+\gamma$ and $\lambda_1 \geq \frac{\gamma}{1+\gamma}$, cases 1 and 2 are feasible: If $\lambda_1+\gamma\mu_1 \geq \gamma$ in addition, PAM is optimal in both cases. On the other hand, if $\lambda_1 + \gamma\mu_1 < \gamma$, then either PAM or the matching rule specified in (25) is optimal. Comparison of social welfare associated with each rule shows that PAM is optimal. If $\lambda < \frac{\gamma}{1+\gamma}$ and $\lambda_1 + \gamma\mu_1 \geq \gamma$, then cases 2 and 3 are feasible: PAM is optimal in case 2 and RM is optimal in case 3. Comparison of social welfare in each case shows that PAM is optimal. If $\lambda < \frac{\gamma}{1+\gamma}$ and $\lambda_1 + \gamma\mu_1 < \gamma$, then only case 3 is feasible and RM is optimal. ■

Proof of Proposition 5.1. Since the expected payoff of a type α_2 seller in the trading game equals zero regardless of his report, and since $t_A(\alpha_1) = t_A(\alpha_2) = 0$, the IC and IR conditions of type α_1 always hold with equality. Let

$$k_1 = \frac{\gamma - t_B(\beta_1)}{1 + \gamma - t_B(\beta_1)} \quad \text{and} \quad k_2 = \frac{\gamma - t_B(\beta_2)}{1 + \gamma - t_B(\beta_2)}.$$ [26]

The optimal bid for a type β_2 buyer equals α_1 if $y = p_A(\alpha_1 \mid \beta_2) \geq k_2$ when he reports his type truthfully, and if $x = p_A(\alpha_1 \mid \beta_1) \geq k_1$ when he misreports his type. Note also that

$$\begin{aligned} x \leq k_1 \quad &\Leftrightarrow \quad t_B(\beta_1) \leq \gamma - \tfrac{x}{1-x}, \\ y \leq k_2 \quad &\Leftrightarrow \quad t_B(\beta_2) \leq \gamma - \tfrac{y}{1-y}. \end{aligned}$$

For any values of x and y, the IC and IR conditions of a type β_1 buyer are given by:

$$x\{\gamma - t_B(\beta_1)\} \geq \max\{0,\ y\{\gamma - t_B(\beta_2)\}\}. \tag{26}$$

[26] Let $k_2 = 0$ if $1 + \gamma - t_B(\beta_2) = 0$.

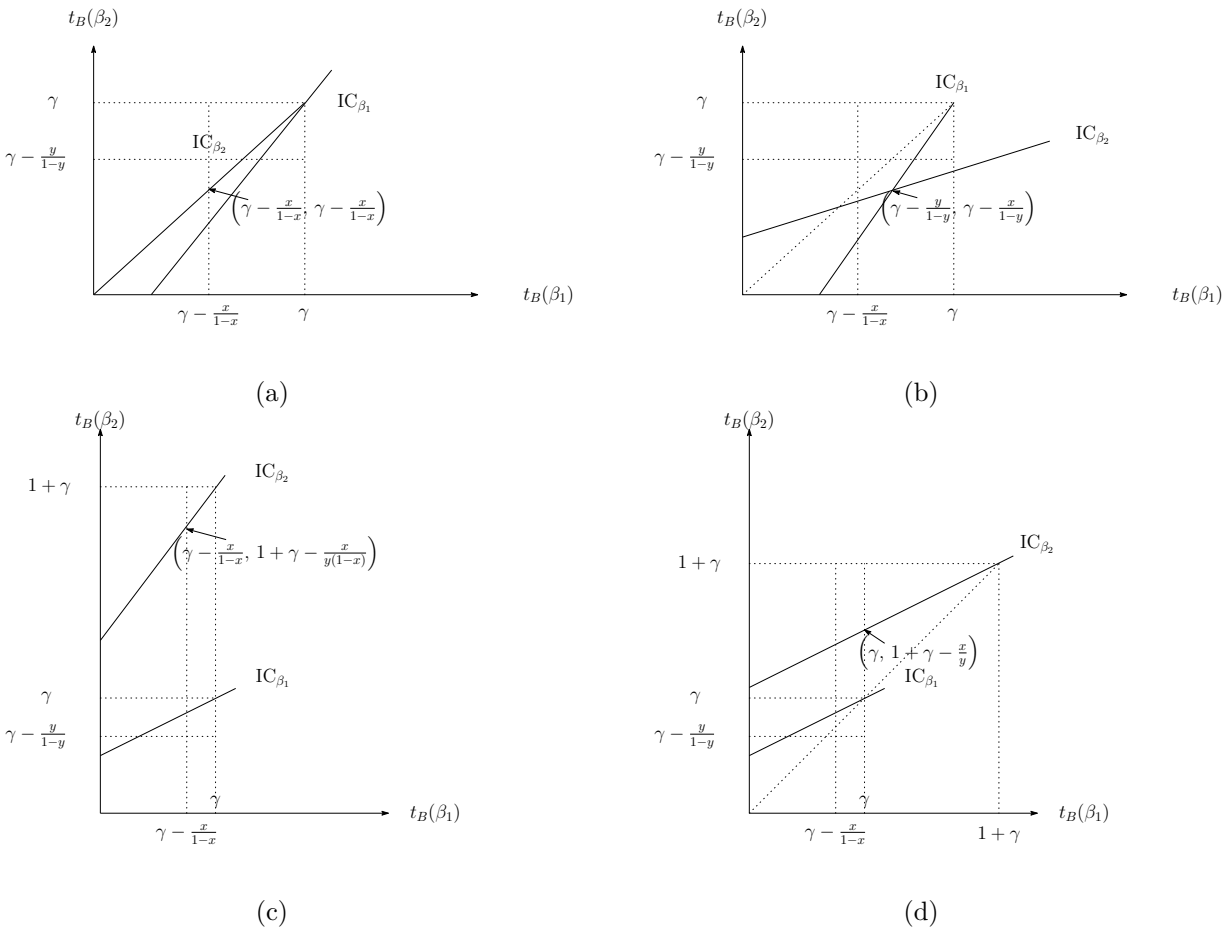

Figure 10: Optimal transfer $(t_B(\beta_1), t_B(\beta_2))$: (a) $x \leq k_1$, $y \leq k_2$, (b) $x > k_1$, $y \leq k_2$, (c) $x \leq k_1$, $y > k_2$, (d) $x > k_1$, $y > k_2$.

a) A type β_2 buyer optimally bids α_2 whether he has reported his type truthfully or not. This requires $x \leq k_1$ and $y \leq k_2$. The IC and IR conditions of type β_2 are given by

$$\gamma - t_B(\beta_2) \geq \max\{0,\, \gamma - t_B(\beta_1)\}. \tag{27}$$

(26) and (27) together show that $y \leq x$. As seen in Figure 10, the optimal transfer in this case is given by

$$t_B(\beta_1) = t_B(\beta_2) = \gamma - \frac{x}{1-x}.$$

On the other hand, the IC and IR conditions of a type α_1 seller are given by

$$z(1 - t_A(\alpha_1)) \geq \max\{0,\, w(1 - t_A(\alpha_2))\}.$$

Substitution of $t_A(\alpha_1) = t_A(\alpha_2) = 0$ yields $z \geq w$, which in turn leads to

$$p_B(\beta_2 \mid \alpha_1) \geq p_B(\beta_2 \mid \alpha_2) \Leftrightarrow \frac{\mu_2}{\lambda_1} p_A(\alpha_1 \mid \beta_2) \geq \frac{\mu_2}{\lambda_2} p_A(\alpha_2 \mid \beta_2) \Leftrightarrow y \geq \lambda_1.$$

Along with $x \geq y$ above and Bayesian plausibility, this implies RM: $x = y = \lambda_1$. It follows that the maximized revenue of the platform in this case is given by

$$R^* = \mu_1 x t_B(\beta_1) + \mu_2 t_B(\beta_2) = (\lambda_1\mu_1 + \mu_2)\left(\gamma - \frac{\lambda_1}{1-\lambda_1}\right).$$

b) A type β_2 buyer optimally bids α_2 when he has reported his type truthfully, but α_1 when he has misreported his type. This requires $x \geq k_1$ and $y \leq k_2$. The IC and IR conditions of type β_2 are given by

$$\gamma - t_B(\beta_2) \geq \max\{0,\, x\{1 + \gamma - t_B(\beta_1)\}\}. \tag{28}$$

Since $t_B(\beta_1) \geq \gamma - \frac{x}{1-x}$, $t_B(\beta_2) \leq \gamma - \frac{y}{1-y}$, and $y t_B(\beta_2) - x t_B(\beta_1) \geq \gamma(y - x)$ by (26), a feasible transfer $(t_B(\beta_1), t_B(\beta_2))$ exists only if

$$y\left(\gamma - \frac{y}{1-y}\right) - x\left(\gamma - \frac{x}{1-x}\right) \geq \gamma(y-x) \quad \Leftrightarrow \quad (y-x)\{1 - (1-x)(1-y)\} \leq 0.$$

Since $(1-x)(1-y) < 1$ by (5), we must have $y \leq x$. In this case, the optimal transfer is given by $(t_B(\beta_1), t_B(\beta_2)) = \left(\gamma - \frac{y}{1-y},\, \gamma - \frac{x}{1-y}\right)$, which satisfies the IC conditions of β_1 and β_2 in (26) and (28) with equality. It also satisfies the IR conditions of both types, as well as $t_B(\beta_1) > \gamma - \frac{x}{1-x}$ and $t_B(\beta_2) \leq \gamma - \frac{y}{1-y}$.

On the other hand, the IC and IR conditions of a type α_1 seller are the same as in case (a), and reduce to $z \geq w$. This coupled with $x \geq y$ implies RM: $x = y = \lambda_1$. It follows that the maximized revenue of the platform in this case is again given by

$$R^* = \mu_1 x t_B(\beta_1) + \mu_2 t_B(\beta_2) = (\lambda_1\mu_1 + \mu_2)\left(\gamma - \frac{\lambda_1}{1-\lambda_1}\right).$$

c) A type β_2 buyer optimally bids α_1 when he has reported his type truthfully, but α_2 when he has misreported his type. This requires $x \leq k_1$ and $y \geq k_2$. The IC and IR conditions of β_2 are given by

$$y\{1 + \gamma - t_B(\beta_2)\} \geq \max\{0,\, \gamma - t_B(\beta_1)\}. \tag{29}$$

Since $t_B(\beta_1) \leq \gamma - \frac{x}{1-x}$, $t_B(\beta_2) \geq \gamma - \frac{y}{1-y}$, and $yt_B(\beta_2) - t_B(\beta_1) \leq y(1+\gamma) - \gamma$ by (29), a feasible transfer $(t_B(\beta_1), t_B(\beta_2))$ exists only if

$$y\Big(\gamma - \frac{y}{1-y}\Big) - \Big(\gamma - \frac{x}{1-x}\Big) \leq y(1+\gamma) - \gamma \quad \Leftrightarrow \quad x \leq y.$$

In this case, the optimal transfer is given by $(t_B(\beta_1), t_B(\beta_2)) = \Big(\gamma - \frac{x}{1-x},\, 1+\gamma - \frac{x}{y(1-x)}\Big)$, which satisfies type β_2's IC condition with equality and also $x = k_1$. On the other hand, the IC and IR conditions of a type α_1 seller always hold with equality. Substituting $y = \frac{\lambda_1 - \mu_1 x}{\mu_2}$ from (5), we can write the platform's expected revenue as:

$$R(\tilde{\Gamma}) = \mu_1 x \left(\gamma - \frac{x}{1-x}\right) + \mu_2 \left(\frac{\lambda_1 - \mu_1 x}{\mu_2}\right) \left(1 + \gamma - \frac{\mu_2 x}{(\lambda_1 - \mu_1 x)(1-x)}\right),$$

which is strictly decreasing in x. This implies that the optimal matching rule in this case is PAM, and the maximized revenue is given by

$$R^* = \begin{cases} \lambda_1(1+\gamma) & \text{if } \frac{\lambda_1}{\mu_2} \leq 1, \\ \lambda_1(1+\gamma) - \frac{\lambda_1 - \mu_2}{\lambda_2} & \text{if } \frac{\lambda_1}{\mu_2} > 1. \end{cases}$$

d) A type β_2 buyer optimally bids α_1 whether he has reported his type truthfully or not. This requires $x \geq k_1$ and $y \geq k_2$. The IC and IR conditions of β_2 are given by

$$y\{1 + \gamma - t_B(\beta_2)\} \geq \max\{0,\, x\{1 + \gamma - t_B(\beta_1)\}\}. \tag{30}$$

(26) and (30) together imply

$$(y - x)\gamma \leq yt_B(\beta_2) - xt_B(\beta_1) \leq (y - x)(1 + \gamma),$$

so that $y \geq x$. In this case, the optimal transfer is given by $(t_B(\beta_1), t_B(\beta_2)) = \Big(\gamma,\, 1 + \gamma - \frac{x}{y}\Big)$, which satisfies type β_2's IC condition and type β_1's IR condition both with equality. On the other hand, the IC and IR conditions of a type α_1 seller always hold. The expected revenue of the platform then equals

$$R(\tilde{\Gamma}) = \mu_1 x t_B(\beta_1) + \mu_2 y t_B(\beta_2) = \mu_2(y - x)(1 + \gamma) + x\gamma.$$

By substituting $y = -\frac{\mu_1}{\mu_2}x + \frac{\lambda_1}{\mu_2}$, we can rewrite this as

$$R(\tilde{\Gamma}) = (1 + \gamma)\left\{\frac{\gamma}{1 + \gamma} - 1\right\} x + \lambda_1(\beta_2 - \alpha_1), \tag{31}$$

which is a decreasing function of x. Hence, the optimal matching rule is PAM, and the maximized revenue is given by

$$R^* = \begin{cases} \lambda_1(1+\gamma) & \text{if } \frac{\lambda_1}{\mu_2} \leq 1, \\ \lambda_1(1+\gamma) - \frac{\lambda_1 - \mu_2}{\mu_1} & \text{if } \frac{\lambda_1}{\mu_2} > 1. \end{cases}$$

Comparison of the maximized revenue in the above four cases shows that the optimal mechanism $\tilde{\Gamma}$ is one described in case (d), which entails PAM, and transfer given by

$$(t_B(\beta_1), t_B(\beta_2)) = \begin{cases} (\gamma,\, 1+\gamma) & \text{if } \frac{\lambda_1}{\mu_2} \le 1, \\ \left(\gamma,\, 1+\gamma - \frac{\lambda_1-\mu_2}{\mu_1}\right) & \text{if } \frac{\lambda_1}{\mu_2} > 1. \end{cases}$$

This mechanism induces a type β_2 buyer to bid α_1 after both truthful and untruthful reporting, and yields the expected revenue as described in (12). The last claim of the proposition on the comparison between $R(\Gamma)$ and $R(\tilde{\Gamma})$ is established as follows.

1. $(1+\gamma)\lambda_1 + \mu_1 > 1+\gamma$: Γ entails PAM.

$$R(\Gamma) = \lambda_1(1+\gamma) - \frac{\lambda_1 - \mu_2}{\mu_1} = R(\tilde{\Gamma}).$$

2. $\lambda_1 + \mu_1 > 1$, $(1+\gamma)\lambda_1 + \mu_1 \le 1+\gamma$ and $\mu_1 > \frac{\gamma}{1+\gamma}$: Γ entails PAM.

$$R(\Gamma) = \frac{\lambda_1 - \mu_2}{\mu_1}\gamma + \mu_2 < R(\tilde{\Gamma}) = \lambda_1(1+\gamma) - \frac{\lambda_1 - \mu_2}{\mu_1}.$$

3. $\lambda_1+\mu_1 > 1$, $(1+\gamma)\lambda_1+\mu_1 \le 1+\gamma$ and $\mu_1 \le \frac{\gamma}{1+\gamma}$: Γ entails BSM (and PAM if $(1+\gamma)\lambda_1+\mu_1 = 1+\gamma$).

$$R(\Gamma) = \lambda_1(1+\gamma) - \frac{\gamma}{1+\gamma} \le R(\tilde{\Gamma}) = \lambda_1(1+\gamma) - \frac{\lambda_1 - \mu_2}{\mu_1},$$

   where the equality holds if and only if $(1+\gamma)\lambda_1 + \mu_1 = 1+\gamma$.

4. $\lambda_1 + \mu_1 \le 1$, $\lambda_1 > \frac{\gamma}{1+\gamma}$, and $\mu_1 \le \frac{\gamma}{1+\gamma}$: Γ entails BSM.

$$R(\Gamma) = \lambda_1(1+\gamma) - \frac{\gamma}{1+\gamma} < R(\tilde{\Gamma}) = \lambda_1(1+\gamma).$$

5. $\lambda_1 + \mu_1 \le 1$, $\lambda_1 + \gamma\mu_1 > \gamma$, and $\mu_1 > \frac{\gamma}{1+\gamma}$: Γ entails PAM.

$$R(\Gamma) = \lambda_1(1+\gamma) - \mu_2\gamma < R(\tilde{\Gamma}) = \lambda_1(1+\gamma).$$

6. $\lambda_1 \le \frac{\gamma}{1+\gamma}$, and $\lambda_1 + \gamma\mu_1 \le \gamma$: Γ entails RM.

$$R(\Gamma) = \lambda_1\gamma < R(\tilde{\Gamma}) = \lambda_1(1+\gamma).$$

■

Proof of Proposition 6.1. Write $\delta_{ij} = \delta(\alpha_i, \beta_j)$ for simplicity. Let

$$q_A(\alpha_i) = \sum_{j=1}^{2} p_B(\beta_j \mid \alpha_i)\,\delta_{ij}, \quad \text{and} \quad q_B(\beta_j) = \sum_{i=1}^{2} p_A(\alpha_i \mid \beta_j)\,\delta_{ij}. \tag{32}$$

$q_A(\alpha_i)$ and $q_B(\beta_j)$ are the expected probabilities of a transaction for a type α_i seller and a type β_j buyer, respectively. The incentive compatibility and individual rationality conditions of each seller type are given by

$$q_A(\alpha_1)\,(-\alpha_1) - t_A(\alpha_1) \geq \max\Big\{q_A(\alpha_2)\,(-\alpha_1) - t_A(\alpha_2),\, 0\Big\},$$
$$q_A(\alpha_2)\,(-\alpha_2) - t_A(\alpha_2) \geq \max\Big\{q_A(\alpha_1)\,(-\alpha_2) - t_A(\alpha_1),\, 0\Big\},$$

and the corresponding conditions of each buyer type are given by

$$q_B(\beta_1)\,\beta_1 - t_B(\beta_1) \geq \max\Big\{q_B(\beta_2)\,\beta_1 - t_B(\beta_2),\, 0\Big\},$$
$$q_B(\beta_2)\,\beta_2 - t_B(\beta_2) \geq \max\Big\{q_B(\beta_1)\,\beta_2 - t_B(\beta_1),\, 0\Big\}.$$

It can be readily verified that incentive compatibility requires $q_A(\alpha_1) \geq q_A(\alpha_2)$ and $q_B(\beta_2) \geq q_B(\beta_1)$, and that under the optimal mechanism, the IR conditions of a type α_2 seller and a type β_1 bind and the IC conditions of a type α_1 seller and a type β_2 buyer bind. It follows that

$$\begin{aligned} t_A(\alpha_2) &= -q_A(\alpha_2)\alpha_2, \quad & t_A(\alpha_1) &= t_A(\alpha_2) - \{q_A(\alpha_1) - q_A(\alpha_2)\}\,\alpha_1, \\ t_B(\beta_1) &= q_B(\beta_1)\beta_1, \quad & t_B(\beta_2) &= t_B(\beta_1) + \{q_B(\beta_2) - q_B(\beta_1)\}\,\beta_2. \end{aligned}$$

The platform's revenue is then given by

$$R(\Gamma^f) = -q_A(\alpha_2)\alpha_2 - \lambda_1\{q_A(\alpha_1) - q_A(\alpha_2)\}\,\alpha_1 + q_B(\beta_1)\beta_1 + \mu_2\{q_B(\beta_2) - q_B(\beta_1)\}\,\beta_2.$$

For $i,\, j = 1,\, 2$, let

$$\eta_{ij} \equiv p_{ij}\delta_{ij}$$

denote the ex ante probability that a transaction takes place between a type α_i seller and a type β_j buyer.

Using η_{ij}, we can rewrite $q_A(\alpha)$ and $q_B(\beta)$ as

$$\begin{aligned} q_A(\alpha_1) &= \frac{1}{\lambda_1}(\eta_{11} + \eta_{12}), q_A(\alpha_2) = \frac{1}{\lambda_2}(\eta_{21} + \eta_{22}), \\ q_B(\beta_1) &= \frac{1}{\mu_1}(\eta_{11} + \eta_{21}), q_B(\beta_2) = \frac{1}{\mu_2}(\eta_{12} + \eta_{22}), \end{aligned}$$

and the revenue from Γ as

$$\begin{aligned} R(\Gamma) &= -q_A(\alpha_2) + q_B(\beta_1) + \mu_2[q_B(\beta_2) - q_B(\beta_1)](1+\gamma) \\ &= -\frac{1}{\lambda_2}(\eta_{21} + \eta_{22}) + \frac{1}{\mu_1}(\eta_{11} + \eta_{21})\gamma + \left\{\eta_{12} + \eta_{22} - \frac{\mu_2}{\mu_1}(\eta_{11} + \eta_{21})\right\}(1+\gamma). \end{aligned}$$

Further simplification yields

$$\begin{aligned} R(\Gamma) &= \frac{1}{\mu_1}\left\{\gamma - (1+\gamma)\mu_2\right\}\eta_{11} + (1+\gamma)\eta_{12} \\ &\quad + \left\{-\frac{1}{\lambda_2} + \frac{\gamma}{\mu_1} - (1+\gamma)\,\frac{\mu_2}{\mu_1}\right\}\eta_{21} + \left(-\frac{1}{\lambda_2} + 1 + \gamma\right)\eta_{22} \\ &= \frac{(1+\gamma)\mu_1 - 1}{\mu_1}\,\eta_{11} + (1+\gamma)\eta_{12} \\ &\quad + \left\{-\frac{1}{\lambda_2} - \frac{1}{\mu_1} + (1+\gamma)\right\}\eta_{21} + \frac{\gamma - (1+\gamma)\lambda_1}{1-\lambda_1}\eta_{22}. \end{aligned}$$

Maximization of R implies $\eta_{12} = p_{12}$, and also $\eta_{21} = 0$ since

$$-\frac{1}{\lambda_2} - \frac{1}{\mu_1} + (1+\gamma) < -2 + (1+\gamma) < 0.$$

Hence,

$$R(\Gamma) = \frac{(1+\gamma)\mu_1 - 1}{\mu_1}\,\eta_{11} + (1+\gamma)p_{12} + \frac{\gamma - (1+\gamma)\lambda_1}{1-\lambda_1}\,\eta_{22}.$$

We consider the following four cases separately depending on the values of λ_1 and μ_1 noting

$$p_{12} = \lambda_1 - p_{11} \text{ and } p_{22} = \lambda_2 - \mu_1 + p_{11}.$$

1. $\mu_1 > \frac{1}{1+\gamma}$ and $\lambda_1 \le \frac{\gamma}{1+\gamma}$.

   Since the coefficients of η_{11} and η_{22} are both positive, we have $\eta_{11} = p_{11}$ and $\eta_{22} = p_{22}$, and can express R in terms of p_{11} as

$$\begin{aligned} R(\Gamma) &= \frac{(1+\gamma)\mu_1 - 1}{\mu_1}\,p_{11} + (1+\gamma)(\lambda_1 - p_{11}) + \frac{\gamma - (1+\gamma)\lambda_1}{1-\lambda_1}\,(\lambda_2 - \mu_1 + p_{11}) \\ &= \frac{(1+\gamma)\mu_1 - 1}{\mu_1}\,p_{11} + \frac{-1}{1-\lambda_1}\,p_{11} + (1+\gamma)\lambda_1 + \frac{\gamma - (1+\gamma)\lambda_1}{1-\lambda_1}\,(\lambda_2 - \mu_1) \\ &= \left(1 + \gamma - \frac{1}{\mu_1} - \frac{1}{\lambda_2}\right) p_{11} + (1+\gamma)\lambda_1 + \frac{\gamma - (1+\gamma)\lambda_1}{1-\lambda_1}\,(\lambda_2 - \mu_1). \end{aligned}$$

   Since

$$1 + \gamma - \frac{1}{\mu_1} - \frac{1}{\lambda_2} < 0,$$

   $p_{11} = 0$ if $\lambda_2 - \mu_1 \ge 0$ and $p_{11} = \mu_1 - \lambda_2$ if $\lambda_2 - \mu_1 < 0$. Note also that

$$(1+\gamma)\lambda_1 + \frac{\gamma - (1+\gamma)\lambda_1}{1-\lambda_1}\,(\lambda_2 - \mu_1) = -\frac{\lambda_2 - \mu_1}{\lambda_2} + (1+\gamma)\mu_2.$$

   Hence, the optimal matching and allocation rules are given by

$$(p_{11}, p_{12}, p_{21}, p_{22}) = \begin{cases} (0, \lambda_1, \mu_1, \lambda_2 - \mu_1) & \text{if } \lambda_1 \le \mu_2, \\ (\mu_1 - \lambda_2, \mu_2, \lambda_2, 0) & \text{if } \lambda_1 > \mu_2, \end{cases}$$

   and

$$(\delta_{11}, \delta_{12}, \delta_{21}, \delta_{22}) = \begin{cases} (*, 1, 0, 1) & \text{if } \lambda_1 \le \mu_2, \\ (1, 1, 0, *) & \text{if } \lambda_1 > \mu_2, \end{cases}$$

   where $*$ can be any number between 0 and 1.

2. $\mu_1 > \frac{1}{1+\gamma}$ and $\lambda_1 > \frac{\gamma}{1+\gamma}$.

   We have $\eta_{22} = 0$ and $\eta_{11} = p_{11}$, and can express R in terms of p_{11} as

$$\begin{aligned} R(\Gamma) &= \frac{(1+\gamma)\mu_1 - 1}{\mu_1}\,p_{11} + (1+\gamma)(\lambda_1 - p_{11}) \\ &= \frac{-1}{\mu_1}\,p_{11} + (1+\gamma)\lambda_1. \end{aligned}$$

Since $\mu_1 > \frac{1}{1+\gamma}$ and $\lambda_1 > \frac{\gamma}{1+\gamma}$, we have $\lambda_2 - \mu_1 < 0$ so that $p_{11} = \mu_1 - \lambda_2$. Hence, the optimal matching and allocation rules are given by

$$(p_{11}, p_{12}, p_{21}, p_{22}) = (\mu_1 - \lambda_2, \mu_2, \lambda_2, 0),$$

and

$$(\delta_{11}, \delta_{12}, \delta_{21}, \delta_{22}) = (1, 1, 0, *),$$

where $*$ can be any number between 0 and 1.

3. $\mu_1 \leq \frac{1}{1+\gamma}$ and $\lambda_1 \leq \frac{\gamma}{1+\gamma}$.

   We have $\eta_{11} = 0$ and $\eta_{22} = p_{22}$, and can express R in terms of p_{11} as

$$\begin{aligned} R(\Gamma) &= (1+\gamma)(\lambda_1 - p_{11}) + \frac{\gamma - (1+\gamma)\lambda_1}{1-\lambda_1}(\lambda_2 - \mu_1 + p_{11}) \\ &= \frac{-(1+\gamma)(1-\lambda_1) + \gamma - (1+\gamma)\lambda_1}{1-\lambda_1} p_{11} + (1+\gamma)\lambda_1 + \frac{\gamma - (1+\gamma)\lambda_1}{1-\lambda_1}(\lambda_2 - \mu_1) \\ &= \frac{-1}{1-\lambda_1} p_{11} + (1+\gamma)\lambda_1 + \frac{\gamma - (1+\gamma)\lambda_1}{1-\lambda_1}(\lambda_2 - \mu_1). \end{aligned}$$

   Since $\mu_1 \leq \frac{1}{1+\gamma}$ and $\lambda_1 \leq \frac{\gamma}{1+\gamma}$, we have $\lambda_2 - \mu_1 \geq 0$ so that $p_{11} = 0$. Hence, the optimal matching and allocation rules are given by

$$(p_{11}, p_{12}, p_{21}, p_{22}) = (0, \lambda_1, \mu_1, \lambda_2 - \mu_1),$$

   and

$$(\delta_{11}, \delta_{12}, \delta_{21}, \delta_{22}) = (*, 1, 0, 1),$$

   where $*$ can be any number between 0 and 1.

4. $\mu_1 \leq \frac{1}{1+\gamma}$ and $\lambda_1 > \frac{\gamma}{1+\gamma}$.

   We have $\eta_{11} = \eta_{22} = 0$ and

$$R(\Gamma) = (1+\gamma)(\lambda_1 - p_{11}).$$

   Hence, $p_{11} = 0$ if $\lambda_2 - \mu_1 \geq 0$ and $p_{11} = \mu_1 - \lambda_2$ if $\lambda_2 - \mu_1 < 0$. Hence, the optimal matching and allocation rules are given by

$$(p_{11}, p_{12}, p_{21}, p_{22}) = \begin{cases} (0, \lambda_1, \mu_1, \lambda_2 - \mu_1) & \text{if } \lambda_1 \leq \mu_2, \\ (\mu_1 - \lambda_2, \mu_2, \lambda_2, 0) & \text{if } \lambda_1 > \mu_2, \end{cases}$$

   and

$$(\delta_{11}, \delta_{12}, \delta_{21}, \delta_{22}) = \begin{cases} (*, 1, 0, 0) & \text{if } \lambda_1 \leq \mu_2, \\ (0, 1, 0, *) & \text{if } \lambda_1 > \mu_2. \end{cases}$$

Combining the four cases, we obtain the conclusion. ■

Proof of Proposition A.2. We show that if RM is optimal with seller-offer bargaining for (λ_1, μ_1), then it is dominated by PAM with buyer-offer bargaining. By Proposition A.1, RM is optimal with seller-offer bargaining when (λ_1, μ_1) satisfies $\mu_1 > \frac{1}{1+\gamma}$ and $\gamma\lambda_1 + \mu_1 > \gamma$, and yields

$$\gamma\mu_2.$$

Furthermore, Figure 6 shows that any such (λ_1, μ_1) satisfies $\lambda_1 \geq 1 - \mu_1 = \mu_2$, and Figure 2 shows that PAM with buyer-offer bargaining is feasible whenever RM is optimal with seller-offer bargaining. By Proposition 4.1, we can evaluate the revenue raised by PAM with buyer-offer bargaining as follows:

1. If $(1+\gamma)\lambda_1 + \mu_1 > 1 + \gamma$, then the revenue equals

$$(1+\gamma)\lambda_1 - \frac{\lambda_1 - \mu_2}{\mu_1} > (1+\gamma)\lambda_1 - \lambda_1 = \gamma\lambda_1 \geq \gamma\mu_2,$$

   where the first inequality follows since $\frac{\lambda_1 - \mu_2}{\mu_1} < \lambda_1 \Leftrightarrow \lambda_1 < 1$.

2. If $\mu_1 > \frac{\gamma}{1+\gamma}$, $\lambda_1 + \gamma\mu_1 > \gamma$, $(1+\gamma)\lambda_1 + \mu_1 \leq 1 + \gamma$, and $\lambda_1 \geq \mu_2$, then the revenue equals

$$\frac{\lambda_1 - \mu_2}{\mu_1} + \mu_2 \geq \mu_2 > \gamma\mu_2.$$

In both cases, hence, PAM with buyer-offer bargaining yields a higher revenue than RM with seller-offer bargaining. A similar argument shows that RM with buyer-offer bargaining is dominated by PAM with seller-offer bargaining. ■

Proof of Proposition A.3.

We will specifically show the following.

- PAM with buyer-offer bargaining if $\frac{1}{1+\gamma} > \lambda_1 > \mu_2$,
- BSM with buyer-offer bargaining if $\lambda_1 > \mu_2 > \frac{1}{1+\gamma}$,
- PAM with seller-offer bargaining if $\frac{1}{1+\gamma} > \mu_2 > \lambda_1$,
- SSM with seller-offer bargaining if $\mu_2 > \lambda_1 > \frac{1}{1+\gamma}$.

1. First fix $d < \frac{1}{1+\gamma}$. If $\|(\lambda_1, \mu_2) - (d, d)\| < \varepsilon$ for a sufficiently small $\varepsilon > 0$, then Figures 2 and 6 show that at (λ_1, μ_1), BSM is optimal with buyer-offer bargaining, and S-squeeze matching is optimal with seller-offer bargaining. The former yields $\lambda_1(1+\gamma) - \frac{\gamma}{1+\gamma}$ in revenue, whereas the latter yields $\mu_2(1+\gamma) - \frac{\gamma}{1+\gamma}$. It follows that BSM with buyer offer is optimal if $\lambda_1 > \mu_2$ and S-squeeze matching with seller offer is optimal if $\lambda_1 < \mu_2$.

2. Next fix $d < \frac{1}{1+\gamma}$. If $\|(\lambda_1, \mu_2) - (d, d)\| < \varepsilon$ for a sufficiently small $\varepsilon > 0$, then Figures 2 and 6 again show that at (λ_1, μ_1), PAM is optimal with both buyer-offer and seller-offer bargaining. If $\lambda_1 > \mu_2$, then buyer-offer yields $\frac{\lambda_1 - \mu_2}{\mu_1}\gamma + \mu_2$ in revenue and seller-offer yields $(1+\gamma)\mu_2 - \lambda_1\gamma$. The former dominates the latter since $\frac{\lambda_1 - \mu_2}{\mu_1}\gamma + \mu_2 > (1+\gamma)\mu_2 - \lambda_1\gamma \Leftrightarrow 1 + \mu_1 > 0$. If $\lambda_1 < \mu_2$, a similar argument shows that PAM with seller-offer bargaining is optimal.

3. Finally, fix $d = \frac{1}{1+\gamma}$ and suppose that $\|(\lambda_1, \mu_2) - (d, d)\| < \varepsilon$ for a sufficiently small $\varepsilon > 0$ and that $\lambda_1 > \mu_2$. If λ_1, $\mu_2 > d$, then we have the same situation as case 1 above. If $\lambda_1 < d$ and $\mu_2 < d$, then we have the same situation as case 2 above. If $\lambda_1 > d$ and $\mu_2 < d$, then S-squeeze matching is optimal with seller-offer bargaining and PAM is optimal with buyer-offer

bargaining. The former yields $\mu_2(1+\gamma) - \frac{\gamma}{1+\gamma}$ in revenue and the latter yields $\frac{\lambda_1-\mu_2}{\mu_1}\gamma + \mu_2$. The latter dominates the former since

$$\begin{aligned}\Big\{\frac{\lambda_1-\mu_2}{\mu_1}\gamma+\mu_2\Big\} - \Big\{\mu_2(1+\gamma) - \frac{\gamma}{1+\gamma}\Big\} &= \gamma\Big\{\frac{\lambda_1-\mu_2}{\mu_1} - \mu_2 + \frac{1}{1+\gamma}\Big\} \\ &> \gamma(d-\mu_2) > 0.\end{aligned}$$

A similar argument proves that PAM with seller-offer bargaining is optimal when $\lambda_1 < \mu_2$.

■

Proof of Proposition A.5. The IC and IR conditions for β_1 are given by

$$x(\beta_1-\zeta) - t_B(\beta_1) \geq \max\{0,\, y(\beta_1-\zeta) - t_B(\beta_2)\},$$

and since $x \geq r$, those for β_2 are given by

$$y(\beta_2-\zeta) - t_B(\beta_2) \geq \max\{0,\, x(\beta_2-\zeta) - t_B(\beta_1)\}.$$

For this to be feasible, we need

$$(y-x)(\beta_1-\zeta) \leq (y-x)(\beta_2-\zeta) \quad \Leftrightarrow \quad y \geq x. \tag{33}$$

We can verify that the IR condition for β_1 and the IC condition for β_2 bind so that $t_B(\beta_1) = x\gamma$ and $t_B(\beta_2) = t_B(\beta_1) + (y-x)\gamma$. On the other hand, the IC and IR conditions for α_1 are given by

$$\zeta - \alpha_1 - t_A(\alpha_1) \geq \max\{0,\, \zeta - \alpha_1 - t_A(\alpha_2)\},$$

and those for α_2 are given by

$$0 - t_A(\alpha_2) \geq \max\{0,\, 0 - t_A(\alpha_2)\}.$$

We obtain from these the optimal transfer function for the seller:

$$t_A(\alpha_1) = t_A(\alpha_2) = 0.$$

It follows that the platform's revenue is given by

$$R = x(\beta_1-\zeta) + \mu_2(y-x)(\beta_2-\zeta) = \{\beta_1-\zeta-\mu_2(\beta_2-\zeta)\}x + \mu_2(\beta_2-\zeta)y. \tag{34}$$

Since this is decreasing in ζ, we set $\zeta = \alpha_1$. Furthermore, comparing the gradient vector of R with the normal vector of the Bayes plausibility condition, we see that PAM is optimal since

$$\frac{\mu_1}{\mu_2} > \frac{\beta_1-\zeta-\mu_2(\beta_2-\zeta)}{\mu_2(\beta_2-\zeta)} \quad \Leftrightarrow \quad \beta_2 > \beta_1. \tag{35}$$

When $\lambda_1 \geq \mu_2$, substitution of $(x,y) = (\frac{\lambda_1-\mu_2}{\mu_1},\, 1)$ yields

$$R^* = \gamma\lambda_1 + \frac{\lambda_2\mu_2}{\mu_1},$$

and when $\lambda_1 < \mu_2$, substitution of $(x, y) = (1, \frac{\lambda_1}{\mu_2})$ yield

$$R^* = \lambda_1(1+\gamma).$$

Note from Proposition 3.1 that when the market is symmetric ($\lambda_1 = \mu_2$), the maximized revenue above equals the maximal social surplus. It follows that this mechanism is optimal among all possible mechanisms under symmetry. ■

Proof of Propositions A.6 and A.7. We will show that the optimal mechanism Γ entails the matching rule p such that

	p_{111}	p_{112}	p_{122}	p_{211}	p_{212}	p_{222}
$0 \le \mu_2 \le \frac{\lambda_2}{2}$	λ_1	0	0	$\lambda_2 - 2\mu_2$	μ_2	0
$\frac{\lambda_2}{2} < \mu_2 \le \frac{1}{2}$	$1-2\mu_2$	$\mu_2 - \frac{\lambda_2}{2}$	0	0	$\frac{\lambda_2}{2}$	0
$\frac{1}{2} < \mu_2 \le 1 - \frac{\lambda_1}{2}$	0	$\frac{\lambda_1}{2}$	0	0	$\mu_1 - \frac{\lambda_1}{2}$	$1-2\mu_1$
$1 - \frac{\lambda_1}{2} < \mu_2 \le 1$	0	μ_1	$\lambda_1 - 2\mu_1$	0	0	λ_2

(36)

Given its simplicity, we prove this first for the second-price auction in which the bidders bid their true values. We then show that the first-price auction yields them exactly the same incentives. In the last step, we prove that the matching rule in (36) has the property that combines NAM between buyers and PAM between a seller and a buyer pair subject to NAM between buyers.

Second-price auction When a buyer bids his true value, his IC and IR conditions are given as follows. For type β_1,

$$0 - t_B(\beta_1) \ge \max\left\{0 - t_B(\beta_2),\, 0\right\}, \tag{37}$$

and for type β_2,

$$\begin{aligned} &\Pr(\alpha_1, \beta_1 \mid \beta_2)\,(v_{12} - v_{11}) + \Pr(\alpha_2, \beta_1 \mid \beta_2)(v_{22} - v_{21}) - t_B(\beta_2) \\ &\ge \max\Big\{\Pr(\alpha_1, \beta_1 \mid \beta_1)(v_{12} - v_{11}) + \Pr(\alpha_2, \beta_1 \mid \beta_1)(v_{22} - v_{21}) - t_B(\beta_1),\, 0\Big\}. \end{aligned} \tag{38}$$

We have from (37) and (38) that

$$\begin{aligned} 0 \le t_B(\beta_2) - t_B(\beta_1) &\le \{\Pr(\alpha_1, \beta_1 \mid \beta_2) - \Pr(\alpha_1, \beta_1 \mid \beta_1)\}\,\Delta_1 \\ &\quad + \{\Pr(\alpha_2, \beta_1 \mid \beta_2) - \Pr(\alpha_2, \beta_1 \mid \beta_1)\}\Delta_2 \\ &= \left(\frac{p_{112}}{\mu_2} - \frac{p_{111}}{\mu_1}\right)\Delta_1 + \left(\frac{p_{212}}{\mu_2} - \frac{p_{211}}{\mu_1}\right)\Delta_2. \end{aligned}$$

For the feasibility of these conditions, we hence need

$$\left(\frac{p_{112}}{\mu_2} - \frac{p_{111}}{\mu_1}\right)\Delta_1 + \left(\frac{p_{212}}{\mu_2} - \frac{p_{211}}{\mu_1}\right)\Delta_2 \ge 0. \tag{39}$$

We can also show that the IR condition for the low type (*i.e.,* β_1) and the IC condition for the high type (*i.e.,* β_2) bind. Hence, when (39) holds, the optimal transfer from the buyer is given by

$$\begin{aligned} &t_B(\beta_1) = 0, \\ &t_B(\beta_2) = \left(\frac{p_{112}}{\mu_2} - \frac{p_{111}}{\mu_1}\right)\Delta_1 + \left(\frac{p_{212}}{\mu_2} - \frac{p_{211}}{\mu_1}\right)\Delta_2. \end{aligned} \tag{40}$$

Turning now to the seller side, recall that their types are observable by the matched buyers. Hence, the incentive compatibility and individual rationality conditions for type α_1 are given by

$$\begin{aligned}&\{1-\Pr(\beta_2,\beta_2\mid\alpha_1)\}v_{11}+\Pr(\beta_2,\beta_2\mid\alpha_1)v_{12}-t_A(\alpha_1)\\&\geq\max\left\{\{1-\Pr(\beta_2,\beta_2\mid\alpha_2)\}v_{11}+\Pr(\beta_2,\beta_2\mid\alpha_2)v_{12}-t_A(\alpha_2),\,0\right\},\end{aligned}\tag{41}$$

and those for type α_2 are given by

$$\begin{aligned}&\{1-\Pr(\beta_2,\beta_2\mid\alpha_2)\}v_{21}+\Pr(\beta_2,\beta_2\mid\alpha_2)v_{22}-t_A(\alpha_2)\\&\geq\max\left\{\{1-\Pr(\beta_2,\beta_2\mid\alpha_1)\}v_{21}+\Pr(\beta_2,\beta_2\mid\alpha_1)v_{22}-t_A(\alpha_1),\,0\right\}.\end{aligned}\tag{42}$$

(41) and (42) together imply

$$\begin{aligned}\{\Pr(\beta_2,\beta_2\mid\alpha_2)-\Pr(\beta_2,\beta_2\mid\alpha_1)\}\Delta_1&\leq t_A(\alpha_2)-t_A(\alpha_1)\\&\leq\{\Pr(\beta_2,\beta_2\mid\alpha_2)-\Pr(\beta_2,\beta_2\mid\alpha_1)\}\Delta_2,\end{aligned}$$

which is equivalent to

$$\left(\frac{p_{222}}{\lambda_2}-\frac{p_{122}}{\lambda_1}\right)\Delta_1\leq t_A(\alpha_2)-t_A(\alpha_1)\leq\left(\frac{p_{222}}{\lambda_2}-\frac{p_{122}}{\lambda_1}\right)\Delta_2.$$

Since $\Delta_2>\Delta_1$ by our assumption (15), this implies that the following feasibility condition must hold:

$$\frac{p_{222}}{\lambda_2}-\frac{p_{122}}{\lambda_1}\geq 0.\tag{43}$$

Again, the IR condition for the low type (*i.e.,* α_1) and the IC condition for the high type (*i.e.,* α_2) bind. Hence, when (43) holds, the optimal transfer from the seller is given by

$$\begin{aligned}t_A(\alpha_1)&=v_{11}+\frac{p_{122}}{\lambda_1}\Delta_1,\\t_A(\alpha_2)&=t_A(\alpha_1)+\left(\frac{p_{222}}{\lambda_2}-\frac{p_{122}}{\lambda_1}\right)\Delta_2.\end{aligned}\tag{44}$$

(40) and (44) yield the maximal payoff for the platform given the matching rule p:

$$\begin{aligned}R(\Gamma)&=\lambda_1t_A(\alpha_1)+\lambda_2t_A(\alpha_2)+2\left\{\mu_1t_B(\beta_1)+\mu_2t_B(\beta_2)\right\}\\&=v_{11}+\frac{p_{122}}{\lambda_1}\Delta_1+\lambda_2\left(\frac{p_{222}}{\lambda_2}-\frac{p_{122}}{\lambda_1}\right)\Delta_2\\&+2\mu_2\left(\frac{p_{112}}{\mu_2}-\frac{p_{111}}{\mu_1}\right)\Delta_1+2\mu_2\left(\frac{p_{212}}{\mu_2}-\frac{p_{211}}{\mu_1}\right)\Delta_2.\end{aligned}\tag{45}$$

The optimal matching rule $p=(p_{111},\ldots,p_{222})$ is one that solves

$$\max\Big\{R(p):\,p\in P\text{ satisfies (39) and (43)}\Big\}.$$

Bayes plausibility (17) allows us to express p_{111}, p_{211} and p_{212} in terms of p_{112}, p_{122} and p_{222} as:

$$\begin{cases}p_{111}=\lambda_1-p_{122}-2p_{112},\\p_{211}=\lambda_2-2\mu_2+2p_{122}+2p_{112}+p_{222},\\p_{212}=\mu_2-p_{222}-p_{122}-p_{112}.\end{cases}\tag{46}$$

We then rewrite the feasibility condition (39) and the platform's payoff (45) in terms of $(p_{112}, p_{122}, p_{222})$:

$$\{\mu_2(\Delta_2 - \Delta_1) + \Delta_2\}\, p_{122} + (1+\mu_2)(\Delta_2 - \Delta_1)\, p_{112} + \Delta_2 p_{222} \le \mu_2\{(\lambda_1 + \mu_2)\Delta_2 - \lambda_1\Delta_1\}, \quad (47)$$

and

$$\begin{aligned} R(\Gamma) = v_{11} &- \frac{2\mu_2}{\mu_1}\lambda_1\Delta_1 + 2\mu_2\left(1 - \frac{\lambda_2 - 2\mu_2}{\mu_1}\right)\Delta_2 \\ &- \left\{\left(\frac{\lambda_2}{\lambda_1} + 2 + \frac{4\mu_2}{\mu_1}\right)\Delta_2 - \left(\frac{1}{\lambda_1} + \frac{2\mu_2}{\mu_1}\right)\Delta_1\right\} p_{122} \\ &- 2\left(1 + \frac{2\mu_2}{\mu_1}\right)(\Delta_2 - \Delta_1)\, p_{112} \\ &- \left(1 + \frac{2\mu_2}{\mu_1}\right)\Delta_2\, p_{222}. \end{aligned}$$

Writing $\kappa = 1 + \frac{2\mu_2}{\mu_1}$, we see that this simplifies to

$$\begin{aligned} R(\Gamma) = v_{11} &- (\kappa - 1)\lambda_1\Delta_1 + 2\mu_2\left(\kappa - \frac{\lambda_2}{\mu_1}\right)\Delta_2 \\ &- \left\{\kappa\Delta_2 + \left(\kappa + \frac{\lambda_2}{\lambda_1}\right)(\Delta_2 - \Delta_1)\right\} p_{122} - 2\kappa(\Delta_2 - \Delta_1)\, p_{112} - \kappa\Delta_2\, p_{222}. \end{aligned} \quad (48)$$

Figure 11 illustrates the feasible combinations of (p_{122}, p_{222}) for $p_{112} < \mu_2 - \frac{\lambda_2}{2}$.

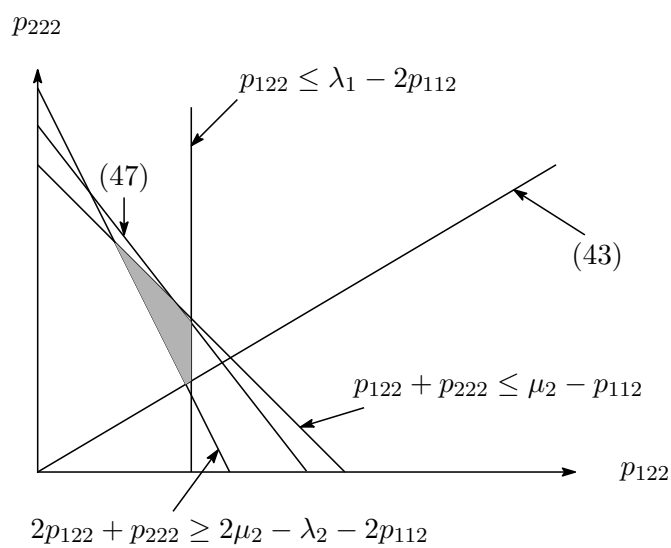

Figure 11: Feasible combinations of (p_{122}, p_{222}) (shaded area) when $p_{112} < \mu_2 - \frac{\lambda_2}{2}$.

$$\begin{aligned} p_{111} \ge 0 \quad &\Leftrightarrow \quad p_{122} \le \lambda_1 - 2p_{112}, \\ p_{211} \ge 0 \quad &\Leftrightarrow \quad 2p_{122} + p_{222} \ge 2\mu_2 - \lambda_2 - 2p_{112}, \\ p_{212} \ge 0 \quad &\Leftrightarrow \quad p_{222} + p_{122} \le \mu_2 - p_{112}. \end{aligned}$$

1. $\mu_2 \le \frac{\lambda_2}{2}$. Let $(p_{112}, p_{122}, p_{222}) = (0, 0, 0)$. It clearly maximizes the platform's payoff (48) subject to $(p_{112}, p_{122}, p_{222}) \ge (0, 0, 0)$. It satisfies (43) and (47) and hence is feasible. By (46), we have

$$(p_{111}, p_{211}, p_{212}) = (\lambda_1,\ \lambda_2 - 2\mu_2,\ \mu_2).$$

2. $\mu_2 > \frac{\lambda_2}{2}$. Since the platform's payoff (48) is decreasing in p_{112}, p_{122} and p_{222}, if $(p_{112}, p_{122}, p_{222})$ is optimal, then it satisfies the constraint $2p_{112} + 2p_{122} + p_{222} \geq 2\mu_2 - \lambda_2 > 0$ with equality. Substituting $p_{222} = 2\mu_2 - \lambda_2 - 2p_{112} - 2p_{122}$ into (48), we obtain

$$\begin{aligned} R(\Gamma) &= v_{11} - \frac{2\mu_2}{\mu_1}\lambda_1\Delta_1 + 2\mu_2\left(1 - \frac{\lambda_2 - 2\mu_2}{\mu_1}\right)\Delta_2 \\ &\quad - \left\{\kappa\Delta_2 + \left(\kappa + \frac{\lambda_2}{\lambda_1}\right)(\Delta_2 - \Delta_1)\right\} p_{122} - 2\kappa(\Delta_2 - \Delta_1)\, p_{112} \\ &\quad - \kappa\Delta_2\,(2\mu_2 - \lambda_2 - 2p_{112} - 2p_{122}) \\ &= v_{11} - \frac{2\mu_2}{\mu_1}\lambda_1\Delta_1 + 2\mu_2\left(1 - \frac{\lambda_2 - 2\mu_2}{\mu_1}\right)\Delta_2 - \kappa\Delta_2(2\mu_2 - \lambda_2) \\ &\quad + \left\{\kappa\Delta_1 - \frac{\lambda_2}{\lambda_1}(\Delta_2 - \Delta_1)\right\} p_{122} + 2\kappa\Delta_1 p_{112}. \end{aligned} \tag{49}$$

There are three subcases to consider.

(a) $\frac{\lambda_2}{2} < \mu_2 \leq \frac{1}{2}$.
Let $(p_{112}, p_{122}) = \left(\mu_2 - \frac{\lambda_2}{2},\, 0\right)$. Since $2\kappa\Delta_1 > \kappa\Delta_1 - \frac{\lambda_2}{\lambda_1}(\Delta_2 - \Delta_1)$, this maximizes (49) subject to the constraints $(p_{112}, p_{122}) \geq (0, 0)$ and $p_{112} + p_{122} \leq \mu_2 - \frac{\lambda_2}{2}$ ($\Leftrightarrow p_{222} \geq 0$). We then have $p_{222} = 2\mu_2 - \lambda_2 - 2p_{112} - 2p_{122} = 0$, and also by (46),

$$(p_{111}, p_{211}, p_{212}) = \left(1 - 2\mu_2,\, 0,\, \frac{\lambda_2}{2}\right).$$

This p clearly satisfies (43). To see that it also satisfies (47), note that

$$\begin{aligned} (47) \quad &\Leftrightarrow \quad (1 + \mu_2)(\Delta_2 - \Delta_1)\left(\mu_2 - \frac{\lambda_2}{2}\right) \leq \mu_2\{\lambda_1(\Delta_2 - \Delta_1) + \mu_2\Delta_2\} \\ &\Leftrightarrow \quad \left\{(1 + \mu_2)\left(\mu_2 - \frac{\lambda_2}{2}\right) - \mu_2\lambda_1\right\}(\Delta_2 - \Delta_1) \leq \mu_2^2\Delta_2 \\ &\Leftarrow \quad (1 + \mu_2)\left(\mu_2 - \frac{\lambda_2}{2}\right) - \mu_2\lambda_1 \leq \mu_2^2 \\ &\Leftrightarrow \quad \mu_2^2 - \frac{\lambda_2\mu_1}{2} \leq \mu_2^2. \end{aligned}$$

(b) $\frac{1}{2} < \mu_2 \leq 1 - \frac{\lambda_1}{2}$. We let $(p_{112}, p_{122}) = (\frac{\lambda_1}{2}, 0)$. This maximizes platform's payoff (49) subject to $2p_{112} + p_{122} \leq \lambda_1$ ($\Leftrightarrow p_{111} \geq 0$). We have $p_{222} = 2\mu_2 - \lambda_2 - 2p_{112} - 2p_{122} = 2\mu_2 - 1$, and hence from (46),

$$(p_{111}, p_{211}, p_{212}) = \left(0,\, 0,\, \mu_1 - \frac{\lambda_1}{2}\right).$$

This p satisfies (43), and also (47) since

$$\begin{aligned}
(47) \quad &\Leftrightarrow \quad (1+\mu_2)\frac{\lambda_1}{2}(\Delta_2-\Delta_1)+\Delta_2(2\mu_2-1) \le \mu_2\left\{\lambda_1(\Delta_2-\Delta_1)+\mu_2\Delta_2\right\} \\
&\Leftrightarrow \quad \left\{\frac{\lambda_1}{2}(1+\mu_2)-\mu_2\lambda_1\right\}(\Delta_2-\Delta_1) \le \Delta_2\mu_1^2 \\
&\Leftrightarrow \quad \frac{\lambda_1}{2}(\Delta_2-\Delta_1) \le \Delta_2\mu_1 \\
&\Leftarrow \quad 1-\frac{\lambda_1}{2} \ge \mu_2.
\end{aligned}$$

(c) $\mu_2 > 1-\frac{\lambda_1}{2}$. Substituting $p_{222} = 2\mu_2-\lambda_2-2p_{112}-2p_{122}$ into the condition $p_{212} \ge 0$ in (46), we obtain $p_{112}+p_{122} \le \mu_2 - p_{222} = -\mu_2+\lambda_2+2p_{112}+2p_{122}$, or equivalently, $p_{122} \ge \mu_2-\lambda_2-p_{112}$. This combined with $p_{122} \le \lambda_1 - p_{112}$ in (46) yields

$$p_{112} \le \mu_1.$$

We let $(p_{112}, p_{122}) = (\mu_1, \lambda_1 - 2\mu_1)$. This maximizes the platform's payoff (49) subject to $2p_{112}+p_{122} \le \lambda_1$ ($\Leftrightarrow p_{111} \ge 0$) and $p_{112} \le \mu_1$. We then have $p_{222} = 2\mu_2-\lambda_2-2p_{112}-2p_{122} = \lambda_2$, and hence from (46),

$$(p_{111}, p_{211}, p_{212}) = (0,\ 0,\ 0)\,.$$

This p satisfies (43) since $\frac{p_{222}}{\lambda_2}-\frac{p_{122}}{\lambda_1} = 1-\frac{\lambda_1-2\mu_1}{\lambda_1} > 0$. To see that it also satisfies (47), note that

$$\begin{aligned}
(47) \quad &\Leftrightarrow \quad \{\mu_2(|Dt_2-\Delta_1)+\Delta_2\}(\lambda_1-2\mu_1)+(1+\mu_2)(\Delta_2-\Delta_1)\mu_1+\Delta_2\lambda_2 \\
&\qquad\quad \le \mu_2\{\lambda_1(\Delta_2-\Delta_1)+\mu_2\Delta_2 \\
&\Leftrightarrow \quad \mu_1^2(\Delta_2-\Delta_1)-\mu_1^2\Delta_2 \le 0.
\end{aligned}$$

This shows that the optimal mechanism Γ with the second-price auction entails the matching rule described in (36).

First-price auction We now turn to the first-price auction. It is useful to analyze the buyers' problem in two interim stages: In the reporting stage, a buyer only knows his own valuation type, whereas in the bidding stage, a buyer also knows the quality of the good sold by the matched seller.

First, consider the bidding stage on the path after truthful reporting by both buyers. The auction game is symmetric between the two buyers since $p_{\alpha 12} = p_{\alpha 21}$ for each α, and hence has a symmetric BNE in which the low valuation buyer (β_1) bids $v_{\alpha 1}$ whereas the high valuation buyer (β_2) chooses his bid according to some distribution $G_\alpha(b)$ with support $[v_{\alpha 1}, \overline{b}_\alpha]$ for some $\overline{b}_\alpha > v_{\alpha 1}$. Call this strategy σ_B. Against σ_B played by the other buyer, when the high valuation buyer β_2 chooses bid $b \in [v_{\alpha 1}, \overline{b}_\alpha]$, his expected payoff is given by

$$(v_{\alpha 2}-b)\left\{\Pr(\beta_1 \mid \alpha, \beta_2)+\Pr(\beta_2 \mid \alpha, \beta_2)G_\alpha(b)\right\}.$$

Since type β_2 is indifferent over bids in the support of G_α,

$$(v_{\alpha 2}-b)\left(\Pr(\beta_1 \mid \alpha, \beta_2)+\Pr(\beta_2 \mid \alpha, \beta_2)G_\alpha(b)\right) = (v_{\alpha 2}-\overline{b}_\alpha). \tag{50}$$

When $b = v_{\alpha 1}$, we have $(v_{\alpha 2} - \bar{b}_\alpha) = (v_{\alpha 2} - v_{\alpha 1}) \Pr(\beta_1 \mid \alpha, \beta_2)$, which yields

$$\bar{b}_\alpha = \Pr(\beta_1 \mid \alpha, \beta_2) v_{\alpha 1} + \Pr(\beta_2 \mid \alpha, \beta_2) v_{\alpha 2},$$

and

$$G_\alpha(b) = \frac{\Pr(\beta_1 \mid \alpha, \beta_2)}{\Pr(\beta_2 \mid \alpha, \beta_2)} \left(\frac{b - v_{\alpha 1}}{v_{\alpha 2} - b} \right) = \frac{p_{\alpha 12}}{p_{\alpha 22}} \left(\frac{b - v_{\alpha 1}}{v_{\alpha 2} - b} \right).$$

Hence, the BNE payoff to the type β_2 buyer in the auction game on the path after truthful reporting equals

$$v_{\alpha 2} - \bar{b}_\alpha = \Pr(\beta_1 \mid \alpha, \beta_2)(v_{\alpha 2} - v_{\alpha 1}). \tag{51}$$

It follows that the type β_2's expected payoff in the reporting stage from truthful reporting equals

$$\begin{aligned} &\Pr(\alpha_1 \mid \beta_2) \Pr(\beta_1 \mid \alpha_1, \beta_2)(v_{12} - v_{11}) + \Pr(\alpha_2 \mid \beta_2) \Pr(\beta_1 \mid \alpha_2, \beta_2)(v_{22} - v_{21}) - t_B(\beta_2) \\ &= \Pr(\alpha_1, \beta_1 \mid \beta_2)(v_{12} - v_{11}) + \Pr(\alpha_2, \beta_1 \mid \beta_2)(v_{22} - v_{21}) - t_B(\beta_2). \end{aligned} \tag{52}$$

Consider now the auction game that follows when a buyer unilaterally misreports his type. If the buyer is the low valuation type (β_1), it is weakly dominant for him to bid $v_{\alpha 1}$, and his expected payoff equals zero. If the buyer is the high valuation type (β_2), his payoff from bidding $b \in [v_{\alpha 1}, \bar{b}_\alpha]$ equals

$$\begin{aligned} &(v_{\alpha 2} - b) \left\{ \Pr(\beta_1 \mid \alpha, \beta_1) + \Pr(\beta_2 \mid \alpha, \beta_1) G_\alpha(b) \right\} \\ &= (v_{\alpha 2} - b) \left\{ \frac{p_{\alpha 11}}{\Pr(\alpha, \beta_1)} + \frac{p_{\alpha 21}}{\Pr(\alpha, \beta_1)} G_\alpha(b) \right\} \\ &= (v_{\alpha 2} - b) \frac{\Pr(\alpha, \beta_2)}{\Pr(\alpha, \beta_1)} \left[\frac{p_{\alpha 11}}{\Pr(\alpha, \beta_2)} + \frac{p_{\alpha 21}}{\Pr(\alpha, \beta_2)} G_\alpha(b) \right] \\ &= \frac{\Pr(\alpha, \beta_2)}{\Pr(\alpha, \beta_1)} (v_{\alpha 2} - \bar{b}_\alpha). \end{aligned}$$

where the last equality follows from (50). Using (51), we can further rewrite this as

$$\frac{\Pr(\alpha, \beta_2)}{\Pr(\alpha, \beta_1)} \Pr(\beta_1 \mid \alpha, \beta_2)(v_{\alpha 2} - v_{\alpha 1}) = \Pr(\beta_1 \mid \alpha, \beta_1)(v_{\alpha 2} - v_{\alpha 1}). \tag{53}$$

Hence, type β_2's expected payoff in the reporting stage from unilateral misreporting is given by

$$\begin{aligned} &\Pr(\alpha_1 \mid \beta_1) \ \Pr(\beta_1 \mid \alpha_1, \beta_1) \, (v_{12} - v_{11}) + \Pr(\alpha_2 \mid \beta_1) \ \Pr(\beta_1 \mid \alpha_2, \beta_1) \, (v_{22} - v_{21}) - t_B(\beta_1) \\ &= \Pr(\alpha_1, \beta_1 \mid \beta_1) \, (v_{12} - v_{11}) + \Pr(\alpha_2, \beta_1 \mid \beta_1) \, (v_{22} - v_{21}) - t_B(\beta_1). \end{aligned} \tag{54}$$

Combining (52) and (54), we see that the IC and IR conditions for type β_2 are just the same as those for the second-price auction. On the other hand, since the expected payoff of type β_1 equals 0 after truthful reporting as well as after misreporting, his IC and IR conditions are again the same as those for the second-price auction given in (37) and (38).

For the checking of the seller's incentive in reporting, we first compute the expected payment by each buyer type in the bidding stage. When the seller is type α, the expected payment by a type β_1 buyer equals

$$\Pr(\beta_1 \mid \alpha, \beta_1) \frac{1}{2} v_{\alpha 1},$$

and that by a type β_2 buyer equals

$$\int_{v_{\alpha 1}}^{\bar{b}_\alpha} b\left[\Pr(\beta_1 \mid \alpha, \beta_2) + \Pr(\beta_2 \mid \alpha, \beta_2) G_\alpha(b)\right] dG_\alpha(b).$$

Using (50) and $(v_{\alpha 2} - \bar{b}_\alpha) = (v_{\alpha 2} - v_{\alpha 1}) \Pr(\beta_1 \mid \alpha, \beta_2)$, we can rewrite this as

$$\begin{aligned}
&\int_{v_{\alpha 1}}^{\bar{b}_\alpha} b\left[\Pr(\beta_1 \mid \alpha, \beta_2) + \Pr(\beta_2 \mid \alpha, \beta_2) G_\alpha(b)\right] dG_\alpha(b) \\
&= v_{\alpha 2} \Pr(\beta_2 \mid \alpha, \beta_2) \int_{v_{\alpha 1}}^{\bar{b}_\alpha} G_\alpha(b) dG_\alpha(b) + \Pr(\beta_1 \mid \alpha, \beta_2) v_{\alpha 1} \\
&= \Pr(\beta_2 \mid \alpha, \beta_2) \frac{1}{2} v_{\alpha 2} + \Pr(\beta_1 \mid \alpha, \beta_2) v_{\alpha 1}.
\end{aligned}$$

Hence, when the type α seller reports his type truthfully, the payment he can expect from a single buyer is

$$\begin{aligned}
&\Pr(\beta_1 \mid \alpha) \Pr(\beta_1 \mid \alpha, \beta_1) \frac{1}{2} v_{\alpha 1} + \Pr(\beta_2 \mid \alpha) \left[\Pr(\beta_2 \mid \alpha, \beta_2) \frac{1}{2} v_{\alpha 2} + \Pr(\beta_1 \mid \alpha, \beta_2) v_{\alpha 1}\right] \\
&= \Pr(\beta_1, \beta_1 \mid \alpha) \frac{1}{2} v_{\alpha 1} + \Pr(\beta_1, \beta_2 \mid \alpha) v_{\alpha 1} + \Pr(\beta_2, \beta_2 \mid \alpha) \frac{1}{2} v_{\alpha 2}.
\end{aligned}$$

The seller's expected revenue from two buyers when he reports his type truthfully is then given by

$$\Pr(\beta_1, \beta_1 \mid \alpha) v_{\alpha 1} + 2 \Pr(\beta_1, \beta_2 \mid \alpha) v_{\alpha 1} + \Pr(\beta_2, \beta_2 \mid \alpha) v_{\alpha 2}.$$

On the other hand, when the seller misreports his type, it will only change the probability that he will be matched with each buyer type since his quality is observed by the buyers. It follows that the seller's IC and IR conditions are just the same as those for the second-price auction given in (41) and (42).

NAM-PAM property of optimal matching The NAM between buyers implies that

$$p(\beta_1 \mid \beta_1) = \begin{cases} 0 & \text{if } \mu_2 \geq \mu_1, \\ 1 - \frac{\mu_2}{\mu_1} & \text{if } \mu_2 < \mu_1, \end{cases} \quad \text{and} \quad p(\beta_2 \mid \beta_2) = \begin{cases} 1 - \frac{\mu_1}{\mu_2} & \text{if } \mu_2 \geq \mu_1, \\ 0 & \text{if } \mu_2 < \mu_1. \end{cases}$$

It follows that

$$p(\beta_1, \beta_1) = \begin{cases} 0 & \text{if } \mu_2 \geq \frac{1}{2}, \\ 1 - 2\mu_2 & \text{if } \mu_2 < \frac{1}{2}, \end{cases} \qquad p(\beta_2, \beta_2) = \begin{cases} 1 - 2\mu_1 & \text{if } \mu_2 \geq \frac{1}{2}, \\ 0 & \text{if } \mu_2 < \frac{1}{2}, \end{cases}$$

and

$$2p(\beta_1, \beta_2) = \begin{cases} 2\mu_1 & \text{if } \mu_2 \geq \frac{1}{2}, \\ 2\mu_2 & \text{if } \mu_2 < \frac{1}{2}. \end{cases}$$

PAM between a seller and a buyer pair then implies the following for the probability of buyer type profiles matched with a high type seller: When $\mu_2 \geq \frac{1}{2}$, $p(\beta_1, \beta_1 \mid \alpha_2) = 0$,

$$2p(\beta_1, \beta_2 \mid \alpha_2) = \begin{cases} 1 - \frac{1 - 2\mu_1}{\lambda_2} & \text{if } 1 - 2\mu_1 < \lambda_2, \\ 0 & \text{if } 1 - 2\mu_1 \geq \lambda_2, \end{cases}$$

and

$$p(\beta_2, \beta_2 \mid \alpha_2) = \begin{cases} \frac{1-2\mu_1}{\lambda_2} & \text{if } 1 - 2\mu_1 < \lambda_2, \\ 1 & \text{if } 1 - 2\mu_1 \geq \lambda_2. \end{cases}$$

On the other hand, when $\mu_2 \leq \frac{1}{2}$, $p(\beta_2, \beta_2 \mid \alpha_2) = 0$,

$$p(\beta_1, \beta_1 \mid \alpha_2) = \begin{cases} 1 - \frac{2\mu_2}{\lambda_2} & \text{if } 2\mu_2 < \lambda_2, \\ 0 & \text{if } 2\mu_2 \geq \lambda_2, \end{cases} \quad \text{and} \quad 2p(\beta_1, \beta_2 \mid \alpha_2) = \begin{cases} \frac{2\mu_2}{\lambda_2} & \text{if } 2\mu_2 < \lambda_2, \\ 1 & \text{if } 2\mu_2 \geq \lambda_2, \end{cases}$$

To summarize, we have

$$(p_{211}, p_{212}, p_{222}) = \begin{cases} (\lambda_2 - 2\mu_2, \mu_2, 0) & \text{if } \mu_2 < \frac{\lambda_2}{2}, \\ (0, \frac{\lambda_2}{2}, 0) & \text{if } \frac{\lambda_2}{2} \leq \mu_2 < \frac{1}{2}, \\ (0, \mu_1 - \frac{\lambda_1}{2}, 1 - 2\mu_1) & \text{if } \frac{1}{2} \leq \mu_2 < 1 - \frac{\lambda_1}{2}, \\ (0, 0, \lambda_2) & \text{if } \mu_2 \geq 1 - \frac{\lambda_1}{2}, \end{cases}$$

Likewise, the probability of buyer type profiles matched with a low type seller (α_1) is given by

$$(p_{111}, p_{112}, p_{122}) = \begin{cases} (\lambda_1, 0, 0) & \text{if } \mu_2 < \frac{\lambda_2}{2}, \\ (1 - 2\mu_2, \mu_2 - \frac{\lambda_2}{2}, 0) & \text{if } \frac{\lambda_2}{2} \leq \mu_2 < \frac{1}{2}, \\ (0, \frac{\lambda_1}{2}, 0) & \text{if } \frac{1}{2} \leq \mu_2 < 1 - \frac{\lambda_1}{2}, \\ (0, \mu_1, \lambda_1 - 2\mu_1) & \text{if } \mu_2 \geq 1 - \frac{\lambda_1}{2}, \end{cases}$$

This p is identical to that described in (36). This completes the proof. ■

Proof of Proposition A.8. Using (46), we can rewrite W in terms of $(p_{112}, p_{122}, p_{222})$ as

$$W(p) = \lambda_1 v_{11} + \lambda_2 v_{21} + 2\mu_2 \Delta_2 - 2(\Delta_2 - \Delta_1)p_{112} - (2\Delta_2 - \Delta_1)p_{122} - \Delta_2 p_{222}. \tag{55}$$

We will identify the socially efficient matching rule p, which solves the following problem.

$$\max_{p_{112}, p_{122}, p_{222}} W(p) \quad \text{subject to} \begin{cases} p_{122} \leq \lambda_1 - 2p_{112}, \\ 2p_{112} + 2p_{122} + p_{222} \geq 2\mu_2 - \lambda_2, \\ p_{112} + p_{122} + p_{222} \leq \mu_2, \\ p_{112},\, p_{122},\, p_{222} \geq 0. \end{cases} \tag{56}$$

As in the proof of Proposition A.6, we proceed by separating cases as follows:

1. $\mu_2 < \frac{\lambda_2}{2}$. Set $(p_{112}, p_{122}, p_{222}) = (0, 0, 0)$. This clearly maximizes $W(p)$ in (55) subject to $(p_{112}, p_{122}, p_{222}) \geq (0, 0, 0)$. We can also verify that it satisfies other constraints in (56). Substituting this back into (46), we obtain $p_{111} = \lambda_1$, $p_{211} = \lambda_2 - 2\mu_2$, and $p_{212} = \mu_2$.

2. $\mu_2 \geq \frac{\lambda_2}{2}$. In this case, the constraint $2p_{112} + 2p_{122} + p_{222} \geq 2\mu_2 - \lambda_2$ should hold with equality since $W(p)$ in (55) is decreasing in the three variables. Hence, we substitute $p_{222} = 2\mu_2 - \lambda_2 - 2p_{112} - 2p_{122}$ into $W(p)$ to rewrite the maximization problem as:

$$\max_{p_{112}, p_{122}} \lambda_1 v_{11} + \lambda_2 v_{22} + \Delta_1(2p_{112} + p_{122}) \quad \text{subject to} \begin{cases} \mu_2 - \lambda_2 \leq p_{112} + p_{122} \leq \mu_2 - \frac{\lambda_2}{2}, \\ 2p_{112} + p_{122} \leq \lambda_1, \\ p_{112},\, p_{122} \geq 0. \end{cases}$$

(a) $\mu_2 \leq \frac{1}{2}$. Since $\mu_2 - \frac{\lambda_2}{2} \leq \frac{\lambda_1}{2}$, the constraint $p_{112} + p_{122} \leq \mu_2 - \frac{\lambda_2}{2}$ holds with equality. The optimal p is then given by $p_{112} = \mu_2 - \frac{\lambda_2}{2}$ and $p_{122} = p_{222} = 0$. Furthermore, $p_{111} = 1 - 2\mu_1$, $p_{211} = 0$ and $p_{212} = \frac{\lambda_2}{2}$.

(b) $\frac{1}{2} < \mu_2 \leq 1 - \frac{\lambda_1}{2}$. If we choose $(p_{112}, p_{122}) = (\frac{\lambda_1}{2}, 0)$, then it maximizes $W(p)$ subject to $2p_{112} + p_{122} = \lambda_1$. It also satisfies the other constraints. Hence, we can take p such that $p_{112} = \frac{\lambda_1}{2}$, $p_{122} = p_{111} = p_{211} = 0$, $p_{212} = \mu_1 - \frac{\lambda_1}{2}$, and $p_{222} = 2\mu_2 - 1$.

(c) $\mu_2 > 1 - \frac{\lambda_1}{2}$. If we choose $(p_{112}, p_{122}) = (\mu_1, \lambda_1 - 2\mu_1)$, then it maximizes $W(p)$ subject to $p_{112} + p_{122} = \mu_2 - \lambda_2$ and $2p_{112} + p_{122} = \lambda_1$. Hence, we can take p such that $p_{112} = \mu_1$, $p_{122} = \lambda_1 - 2\mu_1$, $p_{111} = p_{211} = p_{212} = 0$, and $p_{222} = \lambda_2$.

■