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Uncertain Paternity, Power Utility, and Fractional Moments: The Case of Binomially Distributed Reproductive Success

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Abstract

Newton's Theorem is used derive a formular for the fractional moment of the binomial distribution. The formular is general enough to handle a continuous number of draws and thereby facilitates the analysis of representative agent models where discrete quantities are typically reflected by continuous variables. An application of the formular illustrates that it is easily implemented and can be quickly calculated using standard mathematical software.

Keywords: Uncertain Paternity, Binomial Distribution, Expected Power Utility, Fractional Moment, Newton's Theorem.

JEL Classification: *D10*, *J11*, *J13*.

1 Introduction

Economists seem to have a liking for continuous variables. Even though real world decisions are often discrete by their very nature, economic analyses thereof usually start with writing down the continuous versions of the original decision problems. One stark justification for this approach can be found in the concept of the representative agent because representative agent models aim to reflect and explain average behavior in an economy. Beckerian theories of fertility provide prominent examples where continuous choice frameworks are used to explain discrete real world decisions such as a family's number of kids (Becker, 1991).

Difficulties arise, however, when the variable of interest is of a stochastic nature which typically means that it follows a specific (discrete) probability distribution. As an example, consider the notion of paternal uncertainty which is attracting more and more attention within the economic literature on fertility (see, for example, the papers by Anke Becker

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(2019), Lena Edlund (2013), Brishti Guha (2012), among others). Uncertain paternity occurs when promiscuous behavior lowers the probability of fathering a child below one. As a consequence, the representative man's number of own offspring follows a binomial distribution with two parameters: first the number of his female mating partners and second the aforementioned probability of fatherhood. While the number of mating partners is obviously discrete in nature, the representative agent concept suggests a continuous choice.

In this paper, we will derive an easy to implement formular for the fractional moment of the binomial distribution that is general enough to handle continuous choices of the number of female mating partners.

2 Preferences, Reproductive Technologies, and Fractional Moments

This section briefly sketches a few ingredients of a Beckerian fertility model with paternal uncertainty. Two fundamental asymmetries between women and men are taken into account: asymmetry in offspring recognition and asymmetry in reproductive capacity.[1] The model considers a society that is populated by a large number of single-period lived fertile women and men with equal preferences. Individuals of the same sex are assumed to be homogeneous.

2.1 Female Agents and Limited Reproductive Capacity

The representative woman derives welfare $W_♀$ from reproductive success, which is a function of the (discrete) number $K_♀$ of own biological offspring:

$$W_♀ = V(K_♀) = K_♀^{\mu}, \qquad 0 < \mu < 1 \tag{1}$$

Note that the power utility specification $V(K) = K^{\mu}$ possesses derivatives of all orders. In particular, the function is increasing at a diminishing rate:

$$\frac{\partial^2 V(K)}{\partial K^2} < V(0) = 0 < \frac{\partial V(K)}{\partial K} \tag{2}$$

Moreover, the Arrow Pratt measure of relative risk aversion is constant and equal to $1-\mu$. To account for physiological constraints in female fertility, the reproductive capacity of women is assumed to be bounded by one (see also Willis, 1999):

$$K_♀ \in \{0, 1\} \tag{3}$$

Note that function V measures reproductive success solely in terms of the number of own offspring. This is a simplification compared to the original model in Bethmann and Kvasnicka (2011) where parental investments in child quality play an important role as well. For

[1] For a detailed description of this model please refer to Bethmann and Kvasnicka (2011).

the representative woman, for example, securing additional male child quality contributions provides an incentive to engage in extra-pair mating.

A necessary condition for having a child is that the woman does not abstain from mating. In other words, it requires at least one male mating partner to reproduce:

$$K_{♀} = \min\{1, N_{♀}\} \tag{4}$$

where $N_{♀} \in \mathbb{N}_0$ denotes the number of male mating partners the representative female has chosen. As can be seen from (4), additional male partners do not increase the number of offspring because of limited reproductive capacity. While extra-pair mating might help women to secure (additional) child quality contributions in the original version of the model, we will bypass all child quality decisions as well as mating market considerations in this paper. Here, we simply assume that the representative woman's chosen number of male mating partners ($N_{♀}$) is taken as given by male participants in the mating market.

2.2 Male Agents and Paternal Uncertainty

Like women, men derive utility from reproductive success. Unlike women, however, men are restricted in their reproductive capacity only by access to fertile partners of the opposite sex. They may therefore father more than one child by mating several women. Moreover, unlike women, men do not recognize their offspring. However, men can infer the total number of matings $N_{♀}$ each of their female partners has in equilibrium. For a child born by one of his female partners, a man's probability of fatherhood δ is therefore inversely related to the number of male partners $N_{♀}$ the female has mated, i.e.:

$$\delta = N_{♀}^{-1} \qquad \Leftrightarrow \qquad 1 - \delta = 1 - N_{♀}^{-1} \tag{5}$$

where $1 - \delta$ measures the degree of paternal uncertainty. If women are monogamous, men can be absolutely certain about biological parenthood. If women are promiscuous, however, paternity is uncertain. In this case, the actual but unknown number of own offspring $K_{♂}$ of a man that mates $N_{♂} \in \mathbb{N}_0$ females can be any integer up to and including $N_{♂}$:

$$K_{♂} \in \{0, .., N_{♂}\} \tag{6}$$

Given $N_{♂}$ and δ, the probability of a man to father exactly $K_{♂}$ children therefore follows the binomial distribution with parameters $N_{♂}$ and δ:

$$\mathbb{P}\left(K_{♂}; N_{♂}, \delta\right) = \binom{N_{♂}}{K_{♂}} \left(1 - \delta\right)^{N_{♂} - K_{♂}} \delta^{K_{♂}} \tag{7}$$

Denoting by $\mathbb{E}$ the expectations operator under the above binomial distribution, the expected number of offspring of a man with $N_{♂}$ female partners is hence given by:

$$\mathbb{E}\left[K_{♂}\right] = \delta N_{♂} \tag{8}$$

2.3 Power Utility and Fractional Moments

Because paternity is uncertain, men can only maximize expected reproductive success. Together with the power utility specification this implies that the representative man's welfare $W_♂$ is determined by the (fractional) μ^{th} moment of the binomial distribution with parameters $N_♂$ (number of female partners) and δ (probability of fatherhood):

$$W_♂ = \mathbb{E}\left[V(K_♂)\right] = \mathbb{E}[K_♂^\mu] \tag{9}$$

According to Hoffmann-Jørgensen (1994, p. 303), the fractional moment of a non-negative random variable can be calculated using the following formular:

$$\mathbb{E}[K_♂^\mu] = \frac{\mu}{\Gamma(1-\mu)} \int_0^\infty \frac{1 - M(-z)}{z^{\mu+1}} \,\mathrm{d}z \tag{10}$$

where Γ is Euler's Gamma function[2] and $M(\tau)$ denotes the respective moment generating function with the property that the n^{th} moment about the origin is given by the n^{th} derivative evaluated at zero. The formular (10) can be obtained by using techniques of fractional calculus and taking the derivative of the μ^{th} order of $M(\tau)$ (for details see Wolfe, 1975). For our purposes, it is important to note that $K_♂$ is indeed a non-negative random variable and that the moment generating function of the underlying binomial distribution is given by:

$$M(\tau) = (1 - \delta + \delta \mathrm{e}^\tau)^{N_♂} \tag{11}$$

After some calculus, the substitution of the moment generating function (11) into formular (10) leads us to the following representation of the μ^{th} fractional moment:[3]

$$\mathbb{E}[K_♂^\mu] = \frac{\delta\nu}{\Gamma(1-\mu)} \int_0^\infty \frac{\mathrm{e}^{-t}(1 - \delta + \delta \mathrm{e}^{-t})^{\nu-1}}{t^\mu} \,\mathrm{d}t \tag{12}$$

where $N_♂ \in \mathbb{N}_0$ was replaced with $\nu \geq 1$ in order to stress that we are considering a continuous number of mating partners from now on.[4] Moreover, note that the integral on the right hand side of (12) evaluates to $\Gamma(1-\mu)$ when $\nu = 1$ holds such that $\mathbb{E}[K_♂^\mu] = \delta\nu$ in this case. Hence, we focus on $\nu > 1$ from now on.

3 Fractional Moments of the Binomial Distribution and Newton's Theorem

In this section we will use Newton's generalization of the binomial theorem to rewrite the integrand in (12) in terms of an infinite series. As a quick reminder, let us briefly state how

[2] The Gamma function $\Gamma(\nu) = \int_0^\infty t^{\nu-1}\mathrm{e}^{-t}\,\mathrm{d}t$ extends the factorial function for arbitrary real numbers, except zero and negative integers: $\Gamma(n+1) = n!$ for all $n \in \mathbb{N}_0$. For $\nu \in \mathbb{R}\backslash\{0,-1,-2,..\}$, it satisfies the functional equation $\Gamma(\nu+1) = \nu\Gamma(\nu)$. The incomplete gamma function $\Gamma(\nu, z) = \int_z^\infty t^{\nu-1}\mathrm{e}^{-t}\,\mathrm{d}t$ coincides with the Gamma function when $z = 0$.

[3] The result in equation (12) can be obtained by applying the rule of integration by parts together with an application of L'Hôpital's rule.

[4] The case $N_♂ = 0$ is trivial and of no interest in the current discussion. As argued above, paternity is certain in a monogamous society ($\nu = 1$) and turns uncertain when agents become polygamous ($\nu > 1$).

the binomial theorem for a natural number $a \in \mathbb{N}_0$ expands the polynomial $(x+y)^a$ with $x, y \in \mathbb{R}$ into a finite sum as follows:

$$(x+y)^a = \tbinom{a}{0} x^0 y^a + \tbinom{a}{1} x^1 y^{a-1} + \ldots + \tbinom{a}{a-1} x^{a-1} y^1 + \tbinom{a}{a} x^a y^0 = \sum_{k=0}^{a} \tbinom{a}{k} x^k y^{a-k} \tag{13}$$

As we will see, Newton's generalization by and large resembles the discrete original in (13). However, two small differences have to be discussed. First and foremost, in their most basic version, binomial coefficients $\binom{a}{k} = \frac{a!}{k!\,(a-k)!}$ are defined only for natural numbers $a, k \in \mathbb{N}_0$ with $0 \leq k \leq a$ which is not helpful when considering an arbitrary real number α. Instead, we define the binomial coefficient for arbitrary real numbers $\alpha, \kappa \in \mathbb{R}$ as follows:

$$\binom{\alpha}{\kappa} = \lim_{v \to \alpha} \lim_{w \to \kappa} \frac{\Gamma(v+1)}{\Gamma(w+1)\Gamma(v-w+1)} \tag{14}$$

where the limits occur to handle the singularities of the Gamma function at non-positive integers.[5,6] The second difference refers to the summation in (13). Instead of the finite sum, the generalized binomial formula in the following theorem contains an infinite series.

Theorem (Newton). *Let $x, y \in \mathbb{R}$ where $0 \leq \|x\| < \|y\|$ and let $\alpha \in \mathbb{R}$. Then the expansion of the binomial $(x+y)^\alpha$ is given by an infinite series as follows:*

$$(x+y)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k y^{\alpha-k} \tag{15}$$

where the infinite series in fact converges.

A proof of Newton's Binomial Theorem can be found in many textbooks on real analysis (see, e.g. Protter, 1998). In the following, we will apply the theorem to the expression $(1-\delta+\delta e^{-t})^{\nu-1}$ within the integrand of (12). For this purpose, we must assign the roles of x and y with $\|x\| < \|y\|$. A first attempt could be to set $x = \delta e^{-t}$ and $y = 1-\delta$. Clearly, we can ignore the absolute value operators in this case because both δe^{-t} and $1-\delta$ cannot be negative. However, the t in the exponential function requires our attention. Obviously, the value $t_0 = \log(\frac{\delta}{1-\delta})$ is important in this context because there $1-\delta = \delta e^{-t_0}$ holds. In fact we can distinguish the following cases:

$$t_0 < 0 \quad \Leftrightarrow \quad \delta \in (0, \tfrac{1}{2}) \tag{16a}$$

$$t_0 = 0 \quad \Leftrightarrow \quad \delta = \tfrac{1}{2} \tag{16b}$$

$$t_0 > 0 \quad \Leftrightarrow \quad \delta \in (\tfrac{1}{2}, 1) \tag{16c}$$

[5] In the rest of the paper these limits will play no role so that $\binom{\alpha}{\kappa} = \frac{\Gamma(\alpha+1)}{\Gamma(\kappa+1)\Gamma(\alpha-\kappa+1)}$ always holds.

[6] The appendix to this paper lists a few useful properties of generalized binomial coefficients.

First, we consider the case in (16a) where $\delta \in (0, \frac{1}{2})$. Obviously, $\|x\| = \delta e^{-t} < \frac{1}{2} < 1-\delta = \|y\|$ holds for all $t \in [0, \infty)$ and we apply the binomial theorem (15) with $x = \delta e^{-t}$ and $y = 1-\delta$:

$$\mathbb{E}[K^{\mu}_{\sigma}] = \tfrac{\nu(1-\delta)^{\nu}}{\Gamma(1-\mu)} \int_0^{\infty} \sum_{k=0}^{\infty} \tbinom{\nu-1}{k} \left(\tfrac{\delta}{1-\delta}\right)^{k+1} \tfrac{\mathrm{e}^{-(k+1)t}}{t^{\mu}}\, \mathrm{d}t, \qquad \delta \in (0, \tfrac{1}{2}) \tag{17}$$

Because the sequence of functions $f_k(t) = \binom{\nu-1}{k} \left(\frac{\delta}{1-\delta}\right)^{k} \frac{\mathrm{e}^{-(k+1)t}}{t^{\mu}}$ converges, we may switch integration and summation in (17) by the Fubini-Tonelli Theorem.[7] The subsequent evaluation of the integrals then yields:

$$\mathbb{E}[K^{\mu}_{\sigma}] = \nu(1-\delta)^{\nu} \sum_{k=0}^{\infty} \tbinom{\nu-1}{k} \left(\tfrac{\delta}{1-\delta}\right)^{k+1} \tfrac{1}{(k+1)^{1-\mu}} \tag{18a}$$

Second, we turn to the case in (16b) where δ equals one half such that we can factor out $(\frac{1}{2})^{\nu-1}$ before we set $x = e^{-t}$ and $y = 1$. Obviously, $\|x\| < \|y\|$ holds for all positive real t:[8]

$$\mathbb{E}[K^{\mu}_{\sigma}] = \tfrac{\nu}{\Gamma(1-\mu)} \left(\tfrac{1}{2}\right)^{\nu} \int_0^{\infty} \sum_{k=0}^{\infty} \tbinom{\nu-1}{k} \tfrac{\mathrm{e}^{-(k+1)t}}{t^{\mu}}\, \mathrm{d}t, \qquad \delta = \tfrac{1}{2} \tag{19}$$

Again, the Fubini-Tonelli Theorem allows us to switch integration and summation in (19) and we obtain the following result:

$$\mathbb{E}[K^{\mu}_{\sigma}] = \tfrac{\nu}{2^{\nu}} \sum_{k=0}^{\infty} \tbinom{\nu-1}{k} \tfrac{1}{(k+1)^{1-\mu}} \tag{18b}$$

Third, let us consider the case in (16c) where t_0 is positive. Here, the variable of integration t is decisive when assigning the roles of x and y. Consequently, we split the area of integration and obtain the following equation with two integrals:

$$\mathbb{E}[K^{\mu}_{\sigma}] = \tfrac{\delta\nu}{\Gamma(1-\mu)} \left[\int_0^{t_0} \tfrac{\mathrm{e}^{-t}(1-\delta+\delta\mathrm{e}^{-t})^{\nu-1}}{t^{\mu}}\, \mathrm{d}t + \int_{t_0}^{\infty} \tfrac{\mathrm{e}^{-t}(1-\delta+\delta\mathrm{e}^{-t})^{\nu-1}}{t^{\mu}}\, \mathrm{d}t \right], \qquad \delta \in (\tfrac{1}{2}, 1) \tag{20}$$

When $t \in [0, t_o)$, the inequality $\|\delta e^{-t}\| > \|1-\delta\|$ holds and we set $x = 1-\delta$ and $y = \delta e^{-t}$. Yet, when $t \in (t_0, \infty)$, the reversed inequality $\|\delta e^{-t}\| < \|1-\delta\|$ is true and we set $x = \delta e^{-t}$ and $y = 1-\delta$. In the specific case when $t = t_0$ holds, convergence is also ensured (see also footnote 8) such that a careful application of the binomial theorem yields the following:

$$(1-\delta+\delta\mathrm{e}^{-t})^{\nu-1} = \begin{cases} \delta^{\nu-1} \sum_{k=0}^{\infty} \binom{\nu-1}{k} \left(\frac{1-\delta}{\delta}\right)^{k} e^{(k+1-\nu)t}, & t \in [0, t_0] \\ (1-\delta)^{\nu-1} \sum_{k=0}^{\infty} \binom{\nu-1}{k} \left(\frac{\delta}{1-\delta}\right)^{k} e^{-kt}, & t \in [t_0, \infty) \end{cases} \tag{21}$$

[7]Consider, for example, the series $\sum_{k=0}^{\infty} f_k(t) \leq \sum_{k=0}^{\infty} \|f_k(t)\| < \binom{\nu-1}{\frac{\nu-1}{2}} \frac{1-\delta}{1-2\delta} \frac{\mathrm{e}^{-t}}{t^{\mu}} < \infty$.

[8]Note that the series in (15) converges even when $t = 0$, because we consider a positive exponent $\nu-1 > 0$.

Using this in (20), we can switch the order of summation and integration once again. After evaluating the integrals, we obtain the following result:

$$\mathbb{E}[K_{\sigma}^{\mu}] = \frac{\nu}{\Gamma(1-\mu)} \sum_{k=0}^{\infty} \binom{\nu-1}{k} \Big[\delta^{\nu} \left(\tfrac{1-\delta}{\delta}\right)^{k} \tfrac{\Gamma(1-\mu)-\Gamma(1-\mu,(\nu-k)t_0)}{(\nu-k)^{1-\mu}} + (1-\delta)^{\nu} \left(\tfrac{\delta}{1-\delta}\right)^{k+1} \tfrac{\Gamma(1-\mu,(k+1)t_0)}{(k+1)^{1-\mu}} \Big] \quad (18c)$$

4 Application: Expected Utility when Paternity is Uncertain

In the previous section, we have derived simple formulars for the fractional μ^{th} moment of the binomial distribution with probability of success δ and numbers of trials ν. Restrictions on the parameters were given by: $\mu \in (0, 1)$, $\delta \in (0, 1)$, and $\nu \in [1, \infty)$. The following equation summarizes our results:

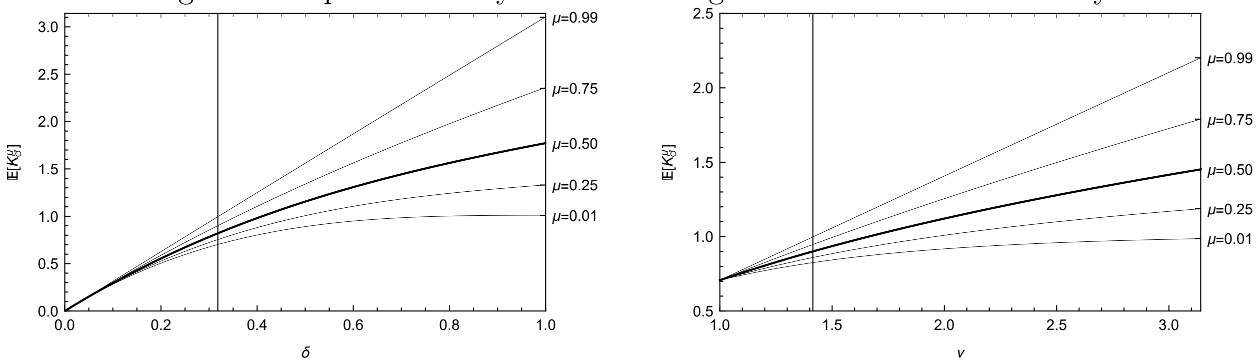
$$\mathbb{E}[K_{\sigma}^{\mu}] = \begin{cases} \nu(1-\delta)^{\nu} \sum\limits_{k=0}^{\infty} \binom{\nu-1}{k} \left(\frac{\delta}{1-\delta}\right)^{k+1} \frac{1}{(k+1)^{1-\mu}}, & \delta \in (0, \frac{1}{2}] \\ \nu(1-\delta)^{\nu} \sum\limits_{k=0}^{\infty} \binom{\nu-1}{k} \left[\left(\frac{\delta}{1-\delta}\right)^{\nu-k} \frac{\Gamma(1-\mu)-\Gamma(1-\mu,(\nu-k)t_0)}{\Gamma(1-\mu)(\nu-k)^{1-\mu}} + \left(\frac{\delta}{1-\delta}\right)^{k+1} \frac{\Gamma(1-\mu,(k+1)t_0)}{\Gamma(1-\mu)(k+1)^{1-\mu}} \right], & \delta \in (\frac{1}{2}, 1) \end{cases} \quad (18)$$

The formular in (18) could be useful in many areas of economic research and related fields. For instance, the binomial distribution plays a prominent role in the field of finance where risk neutrality of agents is still a widely used assumption. Hence, the formula could help to handle risk-averse agents in improved models of the financial markets. In the paper at hand, we stick to the Beckerian theory of fertility where paternity is uncertain. We illustrate the usefulness of the formular in (18) by demonstrating how the fractional moment (and hence expected utility) depends on the various parameters of the model.

According to equation (18), the calculation of the fractional moment can be performed by evaluating an infinite sum where each summand is essentially determined by a binomial coefficient and an expression containing Gamma functions. From a practical point of view the numerical evaluation is not very challenging. Modern computer algebra software packages can efficiently calculate the Gamma function (and its incomplete version) with very high precision. The same is true for generalized binomial coefficients.

As can be seen in Figure 1, the fractional moment $\mathbb{E}[K_{\sigma}^{\mu}]$ becomes a linear function of both the probability of fatherhood as well as the number of female mating partners when the degree of relative risk aversion approaches zero (i.e. μ is close to one). For very low values of μ, utility from reproductive success merely reflects the chance of having own children or not whereas the number of own children is less important (see the left panel of Figure 1). A similar observation can be made when the number of mating partners is increased while keeping the degree of paternal uncertainty constant: a low degree of relative risk aversion renders the utility function into an indicator function for reproductive success that is insensitive to the number own children.

Figure 1: Expected Utility for various degrees of Paternal Uncertainty



NOTES: The thick black lines in both diagrams show the half moment ($\mu = 1/2$) of the binomial distribution with parameters ν and δ. The thin lines display variations of the constant degree of relative risk aversion μ. In the left diagram, different probabilities to father a child $\delta \in (0, 1)$ are considered while setting the number of female mating partners equal to the circular constant $\nu = \pi$. In the right diagram, the number of female mating partners ν varies between one and π while keeping the probability to father a child constant at $\delta = 1/\sqrt{2}$. Intersections with the vertical lines indicate mating market clearing when sex ratios are balanced (i.e. $\delta = 1/\nu$).

5 Conclusion

We have derived an easy to implement formular to calculate fractional moments of the binomial distribution. The formular is general enough to allow for non-integer numbers of draws, a fact that makes our result particularly interesting for representative agent frameworks where discrete real-world quantities are typically modeled by real numbers. Areas where our result might prove useful include the analysis of financial markets with risk-averse agents or macroeconomic models with indivisible labor, to name just a few examples from economics.

To illustrate our result, we have determined the expected utility of a representative male participant in the mating market given his chosen number of mating partners (ν) and given the probability of success (δ). From an individual perspective the number of mating partners is clearly an integer. However, the analysis of average behavior as it is done in a representative agent setting obviously requires a non-integer correspondence. Our result which was summarized in (18) satisfies this requirement. Furthermore, our finding is easily implemented and does not require vast computing power. On an average consumer notebook, it only takes a few seconds to calculate the roughly one thousand fractional moments needed to generate the plots in Figure 1.

Appendix – Useful Properties of Binomial Coefficients

1. General Properties: Let the generalized binomial coefficient be defined as in (14). For all $\alpha \in \mathbb{R}$:

$$\binom{\alpha+1}{k+1} = \binom{\alpha}{k} + \binom{\alpha}{k+1}, \quad k \in \mathbb{Z}; \qquad \binom{\alpha}{k} = \frac{\alpha}{k}\binom{\alpha-1}{k-1}, \quad k \in \mathbb{Z}\backslash\{0\} \tag{22}$$

$$\sum_{k\in\mathbb{Z}} \binom{\alpha}{k} = 2^{\alpha}, \quad \alpha > -1; \qquad \sum_{k\in\mathbb{Z}} (-1)^k \binom{\alpha}{k} = 0, \quad \alpha > 0 \tag{23}$$

$$\sum_{k=0}^{\lfloor \frac{a}{2} \rfloor} \binom{a}{2k} = 2^{a-1}, \quad a \in \mathbb{N}; \qquad \sum_{k=0}^{\lfloor \frac{a-1}{2} \rfloor} \binom{a}{2k+1} = 2^{a-1}, \quad a \in \mathbb{N} \tag{24}$$

See Gould (1972) for these and many other results.
2. Greatest Coefficient: For any upper index $\alpha \in [0, \infty)$ and lower index $\kappa \in [0, \alpha]$ the generalized binomial coefficient $\binom{\alpha}{\kappa}$ is greatest when $\kappa = \frac{\alpha}{2}$.

Proof. The proof is by contradiction. Suppose there is a $\kappa \neq \frac{\alpha}{2}$ such that $\binom{\alpha}{\kappa} \geq \binom{\alpha}{\frac{\alpha}{2}}$ holds. Then by the definition of the generalized binomial coefficient:

$$\log \Gamma(\tfrac{\alpha}{2}+1) \geq \tfrac{1}{2} \log \Gamma(\kappa+1) + \tfrac{1}{2} \log \Gamma(\alpha-\kappa+1) \tag{25}$$

must also hold. However, this inequality contradicts the fact that the gamma function is strictly logarithmically convex which shows that $\binom{\alpha}{\alpha/2}$ is indeed the largest binomial coefficient for non-negative real α and $\kappa \in [0, \alpha]$. □

3. Alternating Signs: For any upper index $\alpha \in [0, \infty)$ and lower index $k \in \mathbb{N}_0$ with $k \geq \lceil \alpha \rceil$ the generalized binomial coefficients are alternately positive and negative as follows:

$$\binom{\alpha}{\alpha+n} = 0, \qquad \forall n \in \mathbb{N} \quad \text{and} \quad \alpha \in \mathbb{N}_0 \tag{26a}$$

$$\binom{\alpha}{\lceil\alpha\rceil+2n} > 0, \qquad \forall n \in \mathbb{N}_0 \quad \text{and} \quad \alpha \notin \mathbb{N}_0 \tag{26b}$$

$$\binom{\alpha}{\lceil\alpha\rceil+2n+1} < 0, \qquad \forall n \in \mathbb{N}_0 \quad \text{and} \quad \alpha \notin \mathbb{N}_0 \tag{26c}$$

Proof. First, we consider the equality in (26a). Evaluating the definition of the generalized binomial coefficient in (14) for α and $\alpha + n$ yields:

$$\binom{\alpha}{\alpha+n} = \frac{\alpha!}{(\alpha+n)!} \lim_{v\to\alpha} \frac{1}{\Gamma(v-\alpha-n+1)} = 0 \tag{27}$$

Note that the argument of the Gamma function in the denominator approaches a non-positive integer. This is where the Gamma function has its poles such that the generalized binomial coefficient is zero in these points.

Second, we consider the inequality in (26b). The proof is by induction. For the base case with $n = 0$, suppose that $\binom{\alpha}{\lceil\alpha\rceil} \leq 0$ holds:

$$\frac{\Gamma(\alpha+1)}{\Gamma(\lceil\alpha\rceil+1)\Gamma(\alpha-\lceil\alpha\rceil+1)} \leq 0 \tag{28}$$

The gamma function evaluated at points $\alpha+1>0$ and $\lceil\alpha\rceil+1>0$ is strictly positive (and finite). Furthermore, $\alpha-\lceil\alpha\rceil+1$ is no integer number such that the gamma function has no singularity in this point. Hence, we obtain the following inequality:

$$\Gamma(\{\alpha\}) = \Gamma(\alpha-\lceil\alpha\rceil+1) \leq 0 \tag{29}$$

This is obviously false because $\{\alpha\}>0$ such that the truth of the base case is verified.

Inductive step $n \mapsto n+1$. Assume the induction hypothesis that $\binom{\alpha}{\lceil\alpha\rceil+2n} > 0$ is true for some $n \in \mathbb{N}_0$. We must show that $\binom{\alpha}{\lceil\alpha\rceil+2n+2} > 0$ holds. Suppose the contrary was true:

$$\binom{\alpha}{\lceil\alpha\rceil+2n+2} \leq 0 \qquad \Leftrightarrow \qquad \binom{\alpha}{\lceil\alpha\rceil+2n}\frac{(\{\alpha\}-2n-2)(\{\alpha\}-2n-1)}{(\lceil\alpha\rceil+2n+1)(\lceil\alpha\rceil+2n+2)} \leq 0 \tag{30}$$

The first term of the last inequality equals the binomial coefficient $\binom{\alpha}{\lceil\alpha\rceil+2n} > 0$ and the denominator of the second term is the product of two strictly positive numbers such that we obtain the following inequality:

$$(\{\alpha\}-2n-2)(\{\alpha\}-2n-1) \leq 0 \tag{31}$$

This statement is false because the product on the left hand side is strictly positive.

Third, we consider the inequality in (26c). The proof is by induction. Contrary to the statement in (26c) suppose that $\binom{\alpha}{\lceil\alpha\rceil+1} \geq 0$ holds for the base case $n = 0$:

$$\frac{\Gamma(\alpha+1)}{\Gamma(\lceil\alpha\rceil+2)\Gamma(\alpha-\lceil\alpha\rceil)} \geq 0 \tag{32}$$

The gamma function evaluated at points $\alpha+1>0$ and $\lceil\alpha\rceil+2>0$ is strictly positive. Furthermore, $\alpha-\lceil\alpha\rceil$ is no integer number and we obtain the following inequality:

$$\Gamma(\{\alpha\}-1) = \Gamma(\alpha-\lceil\alpha\rceil) \geq 0 \tag{33}$$

This statement is obviously false because the gamma function is strictly negative when evaluated at $\{\alpha\}-1$ with $0<\{\alpha\}<1$.

Inductive step $n \mapsto n+1$. Assume the induction hypothesis that $\binom{\alpha}{\lceil\alpha\rceil+2n+1} < 0$ is true for some $n \in \mathbb{N}_0$. We must show that $\binom{\alpha}{\lceil\alpha\rceil+2n+3} < 0$ holds. Suppose the contrary was true:

$$\binom{\alpha}{\lceil\alpha\rceil+2n+3} \geq 0 \quad \Leftrightarrow \quad \binom{\alpha}{\lceil\alpha\rceil+2n+1}\frac{(\{\alpha\}-2n-3)(\{\alpha\}-2n-2)}{(\lceil\alpha\rceil+2n+2)(\lceil\alpha\rceil+2n+3)} \geq 0 \tag{34}$$

The first term of the last inequality equals the binomial coefficient $\binom{\alpha}{\lceil\alpha\rceil+2n+1} < 0$ and the denominator of the second term is the product of two strictly positive numbers such that we obtain the following inequality:

$$(\{\alpha\} - 2n - 3)(\{\alpha\} - 2n - 2) \leq 0 \tag{35}$$

This statement is false because we are looking at a product of two strictly negative numbers. $\square$

4. CONVERGENCE: For any upper index $\alpha \in [0, \infty)$ with $\alpha \notin \mathbb{N}_0$ and lower index $k \in \mathbb{N}_0$ with $k \geq \lceil\alpha\rceil$ the absolute value of the generalized binomial coefficient is declining, that is $\|\binom{\alpha}{k}\| > \|\binom{\alpha}{k+1}\|$.

Proof. The proof is by contradiction. Suppose $\|\binom{\alpha}{k}\| \leq \|\binom{\alpha}{k+1}\|$ holds. Then by the definition of the generalized binomial coefficient:

$$(k - \alpha)\|\Gamma(\alpha - k)\| \geq (k + 1)\|\Gamma(\alpha - k)\| \tag{36}$$

However, this inequality contradicts the fact that α is positive. As a result, we have shown that $\|\binom{\alpha}{k}\| > \|\binom{\alpha}{k+1}\|$ holds for all integers $k \geq \lceil\alpha\rceil$ with $\alpha \in \mathbb{R}_{\geq 0}/\mathbb{N}_0$ as described above. $\square$

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